# Basic Constructive Modality

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ABSTRACT. The benefits of the extended Curry-Howard correspondence relating the simply typed lambda-calculus to proofs of intuitionistic propositional logic and to appropriate classes of categories that model the calculus are widely known. In this paper we show an analogous correspondence between a simple constructive modal logic CK (with both necessity $\Box$ and possibility $\Diamond$ operators) and a lambda-calculus with modality constructors. Then we investigate classes of categorical models for this logic. Parallel work for constructive S4 (CS4) has appeared before in [Bierman and de Paiva, 2000; Alechina *et al.*, 2001]. The work on the basic system CK has appeared initially with co-authors Bellin and Ritter in the conference Methods for the Modalities [Bellin *et al.*, 2001]. Since then the technical work has been improved by [Kakutani, 2007] and taken to a different, higher-order categorical setting by Ritter and myself. Here we expound on the logical significance of the earlier work.

## Preface

It is a great honor to contribute to this volume celebrating Walter Carnielli's work. Professor Carnielli is a source of inspiration and support for people who believe that logic in all its manifestations, pure and applied, is an engine for change and improvement in the sciences and in technology – maybe even in society and in the social sciences, if we, logicians, can get there. With honor comes duty and I have worried about which kind of the technical work would be suitable for this celebration. I have decided to write on "constructive modal logics". Constructive (or intuitionistic, I will use the terms interchangeably) modal logics are modal logics based on a constructive perspective of the world. Constructivists believe in producing witnesses for existential statements; in knowing, given a disjunction $A \vee B$ that one of the disjuncts really holds and that you should know which one is that. They distrust excluded middles and double-negations, which look a bit like "magic" and they really like the notion of implication as a form of internalization of the inferencing process. Constructivists ought to be allowed their own modal logics.

Personally I have been working on constructive modal logics for a long while. I started doing it when I realized that formally the system S4 is just like Linear Logic, the subject of my thesis. Yes, I know that historically this is "back-to-front", the rules for modal S4 must have been in Girard's mind when he conceived Linear Logic, but this was the order that made me interested in modal logic. Work on this formal similarity between the systems with Gavin Bierman eventually became [Bierman and de Paiva, 2000]. Then I discovered the affordances (and intricacies) of formal, explicit substitutions and the preliminary work on a dual system for intuitionistic and (S4) modal logic, called DIML for Dual and Intuitionistic Modal logic (joint with Neil Ghani and Eike Ritter[Ghani *et al.*, 1998]) came to light. By then I was truly bitten by the bug: I wanted to see how far we can push the frontiers of the Curry-Howard correspondence for modal systems. But I wanted my Curry-Howard correspondence to be a categorical one, that is, I wanted "triangles" of maps relating logics, their type-theoretical formulations and their (equivalent) categorical semantics formulations. With the help of Natasha Alechina, Eike Ritter and Michael Mendler I wrote about the relationship between categorical semantics and possible world semantics in [Alechina *et al.*, 2001], but we barely scratched the surface of the question. Paying attention to the philosophical tradition that considered $K$ (named after Kripke[Kripke, 1963]) the basic system for normal modal logic, Gianluigi Bellin, Eike Ritter and myself applied to this system our basic Natural Deduction intuitions in [Bellin *et al.*, 2001].This is the work we discuss here.

Meanwhile I have been helping to organize a collection of workshops on "Intuitionistic Modal Logics and Applications (IMLA)", hoping to get philosophers and computer scientists to share their insights on the big quest for a Curry-Howard-Lawvere[1] correspondence for intuitionistic Modal Logic in general. The IMLA workshops started in 1999, as part of the Federated Logic Conference (FLoC1999) and the fifth installment has just happened as part of the Congress of Logic, Methodology and Philosophy of Science in Nancy, France, 2011. Associated with the IMLA workshops there have been journal special volumes published as [Fairtlough *et al.*, 2001; Goré *et al.*, 2004; de Paiva and Pientka, 2011].

But work on intuitionistic modal logics is still very much in construction, the quest is just beginning, we have some pieces of the puzzle in place, but much remains to be done. It is also a work where more philosophical intuition is required. Mathematics alone can only go so far and hence this is a work where we could do with help from logicians that are well-versed in philosophical questions. This brings us back to Walter Carnielli and his ability of, not only doing first-rate work on his own, straddling philosophical and mathematical fields of expertise, but also of congregating and organizing other logicians to work on interesting problems. Thus this work is dedicated to Walter in the hope that he will like the project of Curry-Howard-Lawvere correspondences for constructive Modal Logic and he will use some of his uncanny abilities to further it. Happy Birthday, Walter!

## 1 Introduction

Modal logic is arguably the logic formalism most used in Computer Science. The *explicit* logic formalism most used, as classical logic is used implicitly everywhere. Modal Logic in its several variants, e.g. epistemic logic, temporal logic, description logics, probabilistic logic, etc.. have found compelling applications in Artificial Intelligence, Knowledge Representation, Verification, Software En-

[1]Some people would call it a Curry-Howard-Lambek correspondence and this would be a good name too.

gineering, etc. The work of the Coalgebraic Modal school, for example as described in [Cirstea *et al.*, 2007], makes a convincing case for modal logic in general and the coalgebraic approach in particular. But while the coalgebraic approach is encompassing, it is not the whole story on the categorical way to modal logic. The coalgebraic approach is based on semantics of modal logics in terms of relational structures and can be seen as following the tradition of categorical 'model' theory. Here we are interested in the categorical 'proof' theory approach to modal logic.

Methods of categorical proof theory are useful both in explaining logic systems and in providing us with possible implementations and applications of those systems. Our guiding intuitions come from the Curry-Howard interpretation and its uses as foundations for Functional Programming. As usual, the categorical proof theory approach to modal logic is less developed than the corresponding modal theoretical one. Our aim is to help balance the issue by providing a basic, but fundamental piece of the proof theoretical approach in detail.

The most basic classical modal logic is usually taken to be system $\mathsf{K}$ (after Kripke), where one has two operators $\Box$ (necessity) and $\Diamond$ (possibility) satisfying the axiom $\Box(A \to B) \to \Box A \to \Box B$ and the necessitation rule. The operator $\Diamond$ is usually taken to be defined in terms of $\Box$ and negation, as $\Diamond A = \neg(\Box(\neg A))$. But categorical proof theory is more transparent over a constructive basis, so we need to have independent definitions for $\Box$ and $\Diamond$ and, as traditional when constructivizing concepts, one is faced with multiple possibilities.

## 2 Which basic constructive modal logic?

It is traditional to face a 'plurality' problem when constructivizing notions. Usually a single notion in classical mathematics gives rise to several possible notions when using constructive logic. When confronted with the problem of defining the intuitionistic or constructive modal system corresponding to the classical modal system $\mathsf{K}$ many different systems present themselves. Rather than choosing one such system and calling it 'the' constructive system corresponding to $\mathsf{K}$ modal logic, we prefer to discuss briefly two such systems and then then concentrate in the one we prefer.

The first constructive system we discuss corresponding to $\mathsf{K}$ was described (together with a whole framework of other constructive modal logics) in Simpson's thesis[Simpson, 1994]. This system (called $\mathsf{IK}$ for intuitionistic $\mathsf{K}$) had independently being proposed earlier by Fisher-Servi and others and satisfies many properties that one might expect of a basic constructive modal logic. These include non-interdefinable modal operators $\Box$ (for necessity) and $\Diamond$ (for possibility). The logical basis of the system is intuitionistic propositional logic ($\mathsf{IPL}$) and adding to the independently defined modalities the law of the excluded middle takes us back to the the traditional system $\mathsf{K}$. In addition the system satisfies a *disjunctive* as well as an *existence* property, as characteristic of intuitionistic logic. One possible axiomatization of the system is presented below.

As it happens in the classical system $\mathsf{K}$, the necessity operator distributes over conjunctions, while symmetrically the possibility operator distributes over disjunctions. The constants follow the same pattern.

$$\Box(A \wedge B) \iff \Box A \wedge \Box B$$
$$\Box(\top) \iff \top$$
$$\Diamond(A \vee B) \iff \Diamond A \vee \Diamond B$$
$$\Diamond(\bot) \iff \bot$$

The system $\mathsf{IK}$ comes from the strong semantic intuition of possible worlds, where we say that $\Box A$ holds in the current world, if $A$ holds in all the worlds accessible from it. And symmetrically, $\Diamond A$ holds in the current world if there exists one world accessible from the current world where $A$ holds.

The second system, called $\mathsf{CK}$ (from constructive $\mathsf{K}$ comes from proof-theoretical intuitions provided by Natural Deduction formulations of logic. It was described in [Bellin *et al.*, 2001], following the adaptation of Prawitz's suggestions in his seminal work [Prawitz, 1965] for $\mathsf{S4_I}$ to $\mathsf{K}$. Like the previous system $\mathsf{IK}$ we have an intuitionistic propositional basis and non-interdefinable operators for necessity and possibility. Like $\mathsf{IK}$ the system satises a disjunctive as well as an existence property, as characteristic of intuitionistic logic. Unlike $\mathsf{IK}$ this system does not warrant distribution of the possibility operator over disjunction. The problem is that in a naive Natural Deduction environment one cannot see how to deduce $\Diamond A \vee \Diamond B$ from $\Diamond(A \vee B)$ and hence this distribution (algebraically very appealing) is left out of the system.

A main advantage of system $\mathsf{CK}$ is that it is easy to define terms in a lambda-calculus corresponding to the operations of $\mathsf{CK}$, as done in [Bellin *et al.*, 2001]. The rules repeated below are not as symmetric as in constructive versions of $\mathsf{S4}$. The introduction of *commuting conversion* rules, necessary to expose $\beta$-redexes, as in the case of $\mathsf{CS4}$ make the term calculus not as streamlined as we would like it to be. However, this is a natural logic to consider, if one is determined to push the frontiers of the Curry-Howard Isomorphism, as far as they will go.

The modal system $\mathsf{CK}$ fewer symmetries than system $\mathsf{CS4}$ make it harder to conform to the Natural Deduction requirements of introduction and elimination pairs of rules, each pair defining a single connective, each pair satisfying local reduction rules. Thus we are faced with a problem of lack of structure, which makes the modelling of $\mathsf{CK}$ more challenging than the modelling of $\mathsf{CS4}$. Meanwhile the system $\mathsf{IK}$ has more algebraic structure, but as it is usually presented does not lend itself well to categorical modelling. The problem is that the kinds of judgements we are modelling ($M$ is a term of type $A$ and $A$ is related to $A'$ via an accessibility relation) are quite different. Despite the lack of overall best system, in this particular work, we prefer system $\mathsf{CK}$ to system $\mathsf{IK}$, as the categorical modelling is our immediate goal.

## 3 The System $\mathsf{CK}$

We aim at a propositional system that, like classical or intuitionistic logic, can be presented either as an axiomatic system, or a sequent calculus or a Natural Deduction system and such that these different presentations are proved 'equivalent', in the sense of, at least, proving the same theorems. (Whether each proof in one system can be transformed or not in a proof of the other system is a harder case, left for future work.) We discuss these different formalisms for our chosen basic constructive necessity system $\mathsf{CK}$.

### 3.1 Sequent Calculus and Axiomatic System

The sequent calculus rules and Hilbert-style axioms for the system $\mathsf{CK}$ are relatively uncontroversial and well-known.

To define a sequent calculus for $\mathsf{CK}$ we add to the sequent calculus rules for intuitionistic propositional logic ($\mathsf{IPL}$) two rules. One rule for the necessity modal operator ($\Box$) and a similar rule for the modality of possibility ($\Diamond$).

$$\frac{\Gamma \vdash B}{\Box\Gamma \vdash \Box B}\,Box \qquad \frac{\Gamma, A \vdash B}{\Box\Gamma, \Diamond A \vdash \ \Diamond B}\,Diamond$$

These rules do double-duty as they work as both left and right introduction rules for the necessity $\Box$ and the possibility $\Diamond$ modal operators. These rules are also slightly awkward in that they are not strictly left or right rules and the rule for $\Diamond$ already mentions the $\Box$ operator. However, the rules are sufficient to prove the necessary syntactic theorems and they do provide us with a cut elimination theorem, as shown, for instance in (in a more complicated form) by Wisejekera in [Wijesekera, 1990].

Similarly to the sequent calculus, we can add to any axiomatization of propositional Intuitionistic Logic ($\mathsf{IPL}$) the following three axioms:

$$\Box(A \to B) \to (\Box A \to \Box B)$$

$$\Box(A \to B) \to (\Diamond A \to \Diamond B)$$

$$\Box A \times \Diamond B \to \Diamond(A \times B)$$

and the Necessitation Rule:

$$\frac{\vdash B}{\vdash \Box B}$$

to obtain an axiomatization of $\mathsf{CK}$. Other axiomatizations are possible, but do not shed much light on the essence of the system. Wijesekera shows that the sequent calculus above corresponds to the axiomatic formulation given by axioms for intuitionistic logic, plus axiom

$$\Box(A \to B) \to (\Box A \to \Box B)$$

together with rules for Modus Ponen and Necessitation:

$$\frac{\vdash A \to B \qquad \vdash A}{\vdash B}\,\mathrm{MP} \qquad \frac{\vdash A}{\vdash \Box A}\,Nec$$

He then proves a Craig interpolation theorem for his system, one of the usual consequences of syntactic cut-elimination.

Wisejekera also produces Kripke, algebraic and topological semantics for this calculus. From our "wish list" for logical systems a natural deduction formulation and a categorical semantics are missing. These we proceed to discuss, in turn.

Prawitz in his classic monograph on Natural Deduction only discusses natural deduction formulations for the basic modal logics S4 and S5. As Bull and Segerberg note in their survey "Basic Modal Logic" in page 27

> [..]It has proved difficult to extend this sort of analysis [the Natural Deduction one] to the great multitude of other systems of modal logic.

According to Bull and Segerberg, some logicians tried to blame the intensional character of modal logic for the poor fit between modal logic and Gentzen proof theoretical methods. But intuitionistic logic is also intensional and Gentzen methods work like a treat for it. Another suggestion is that there is an unreasonable proliferation of modal logics and "natural" deduction methods would only apply to "natural" enough logics. Granting for the moment that this may be the case, it is still strange that sequent calculus systems have been devised for a whole family of modal logics, while natural deduction formulations only exist, traditionally, for S4 and S5.

The problem is easy to see: if one wants to think of the sequent calculus rule for $\Box$ above as applying to natural deduction derivation-trees we have that a tree

$$\begin{array}{c} A_1, A_2, \ldots, A_k \\ \vdots\, \pi \\ B \end{array}$$

has to be transformed into a tree of shape somewhat like

$$\dfrac{\begin{array}{c} \Box A_1, \Box A_2, \ldots, \Box A_k \\ \vdots\, \pi^* \\ B \end{array}}{\Box B}$$

but while transforming the conclusion of a natural deduction derivation tree is perfectly acceptable, modifying its premisses is not allowed, usually. Moreover, if modifying of premisses was allowed, we would still have a problem as the inference

$$\frac{\Box A}{A}$$

perhaps used to get from the 'new' hypotheses $\Box A_i$ back to the given ones $A_i$, is valid in $\mathsf{S4}$, but not in $\mathsf{K}$, the system we want to model.

In the following discussion, as in the previous one ([Bellin *et al.*, 2001]) in which this is based, we present three different solutions for this problem. Neither of the solutions is completely satisfactory, for reasons we discuss in section 6. We feel that discussing the problem and its partial solutions is worthwhile though, hoping that someone will produce better solutions.

# 4 Natural Deduction for CK?

The first "solution" has existed for quite a while. In the 50's Fitch proposed a variant of Gentzen's Natural Deduction, which for classical or intuitionistic logic seems simply a notational variant: one writes linear derivations instead of tree-like derivations. But Fitch-style Natural Deduction and Gentzen-style Natural Deduction are further apart than one might expect. While at the level of classical (or intuitionistic) logic the differences seem only notational, when modal operators are considered the gap seems to widen. Fitting's 1983 monograph "Proof Methods for Modal and Intuitionistic Logics" considers several Fitch-style natural deduction systems for different modal systems built from system K plus axioms. More recently Borghuis (in his doctoral thesis) has developed some of the Fitch-ND systems to devise a Curry-Howard "proofs-as-types"[2] interpretation for several systems of modal logic.

A second solution to the problem can be seen as a corollary of work of Gianluigi Bellin done in the early 80's[Bellin, 1985]. We recap that work briefly and extend it, both with a term assignment system and a categorical model.

Finally a third solution, inspired by work on systems where the context is divided in zones, such as Girard's LU or Plotkin and Barber's DILL[Barber, 1996] is presented. Relating these different solutions is left for future work[3].

Because of our emphasis in categorical modelling, we will concentrate in solutions two and three above. Putting it bluntly I do not know how to extract terms and a categorical semantics from a Fitch-like Natural Deduction formulation of CK. As we will see in sequel, the terms and categorical semantics we do develop still have some issues in need of clarification.

Summarizing: Our aim in this note is two-fold. First we want to develop a term calculus with usual syntactic properties (strong normalization, subject reduction, confluence) for a Gentzen-like (as opposed to Fitch-like) Natural Deduction version of modal system K. Second we would like to extend (Borghuis and our own) work on natural deduction and term assignment systems, with the further correspondence between typed $\lambda$-calculus and category theory, that is usually referred as the "extended" (or categorical) or Curry-Howard-Lawvere correspondence.

## 4.1 Fitch-style Modal Natural Deduction

Borghuis work ([Borghuis, 1994; Borghuis, 1998]) is based on Pure Type Systems, Barendregt's beautiful systematization of work on several (higher-order) lambda-calculi. His framework is very expressive, but since we are not interested (at least in this note) on higher-order systems, some of the complications of Borghuis system can be avoided.

The main idea of Fitch-style ND proofs is that to prove an implication, say $A \to B$, we go into what is called a *subordinate proof* where we assume that

[2]Borghuis system seems strange in that, despite the fact that his basic systems are all constructive, the modal logics on top are classical. This choice seems dictated more by tradition (Barendregt's cube is constructive, while most work on modal logic is classical) than by application suitability.

[3]Recent other approaches (Vigano, Gabbay, Martini& Masini) that tie in the semantics of modal logics (in terms of possible worlds) with their sytanctic presentation have been devised. We will not have much to say about this line of work in this paper.

the antecedent of the proposed implication, $A$ is true. If we can derive $B$ under this assumption, then we can discharge the assumption, by exiting the subordinate proof (which is confusedly also called a *box*) and adding $A \to B$ to the original proof. One is also allowed to reiterate any formulas from the main proof into a subordinate proof, this simply corresponds to many uses of the same assumption. This device makes it possible to write proofs in a linear order, but does not change the intuitive meaning of the implication rule.

Having invented this box for dealing with implication, it is now natural to invent a different kind of box to deal with the modality $\Box$. Fitch adds a new kind of subordinate proof, a *strict subordinate proof* to his system in his book[Fitch, 1952]. A strict subordinate proof requires no hypothesis, and more importantly "reiteration" in a strict subordinate proof is restricted to formulas of a certain form. For the logic we are interested here, system $\mathsf{K}$, one is only allowed to reiterate formulas of the form $\Box A$, ie modal formulas and their reiteration appears in the subordinate proof without the $\Box$ operator., ie as simply $A$. This is the so-called *K-import rule.* A formula imported into a strict subordinate proof does not count as a hypothesis of that proof.

To export proofs from the box, we have to be a bit more careful: a conclusion $A$ can be exported if it was derived by means of a *categorical* strict subordinate proof. (Note that this categorical has nothing to do with Category Theory.) A categorical strict subordinate proof has to satisfy:

1. All of its assumptions have been discharged;

2. the conclusion lies directly inside the modal interval;

3. there are no nested subordinate proofs that are still "open".

We want to simplify Borghuis type theory to deal simply with the *propositional* modal logic K. We also want a *constructive* propositional basis, so we shall not use encodings of operators. We have ground types, implications $A \to A$, conjunctions $A \wedge A$, and modal types $\Box A$. We omit disjunctions and falsum, for the time being.

For modal Fitch-style natural deduction we need to attribute a degree of nestedeness to each formula in each proof. In a non-modal Fitch setting the degree of a formula is simply the number of hypotheses at that stage of the proof. In a modal setting the degree of a formula is a pair of natural numbers $(m, h)$ where $m$ is the modal depth of the formula and $h$ is the number of hypotheses at that stage of the proof.

The following is a simplification (no higher-order logic) of Borghuis system. The type theory has judgements

$$\Gamma_k | \ldots | \Gamma_1 | \Gamma_0 \vdash M \colon A$$

meaning that (logically) $M$ is a proof of $A$ where a context has $k$-many compartments, where $k$ is the maximum modal depth of formulas in the derivation. We have the following raw terms:

$$M ::= x \;\mid\; \lambda a \colon A.M \;\mid\; MM \;\mid\; \mathtt{Box}(M) \;\mid\; \mathtt{Unbox}(M)$$

The typing rules are as follows

$$\frac{}{\Gamma|\Gamma_0, x{:}\,A \vdash x{:}\,A}$$

$$\frac{\Gamma|\Gamma_0, a{:}\,A \vdash M{:}\,B}{\Gamma|\Gamma_0 \vdash \lambda a{:}\,M{:}\,A \to B} \qquad \frac{\Gamma|\Gamma_0 \vdash M{:}\,A \to B \qquad \Gamma|\Gamma_0 \vdash N{:}\,A}{\Gamma|\Gamma_0 \vdash MN{:}\,B}$$

$$\frac{\Gamma|\Gamma_0 \vdash M{:}\,\Box A}{\Gamma|\Gamma_0|_{-} \vdash \mathtt{Unbox}(M){:}\,A} \qquad \frac{\Gamma|_{-} \vdash M{:}\,A}{\Gamma \vdash \mathtt{Box}(M){:}\,\Box A}$$

Usual desirable meta-theoretical results hold for this calculus. In particular Borghuis proved:

THEOREM 1 (Borghuis). *Subject Reduction, Strong Normalization (SN) and the Church-Rosser property hold for the calculus above.*

The proof detailed in Borghuis thesis proceeds by defining an erasing mapping from the modal lambda-calculus to the simply typed $\lambda$-calculus and proving properties of this erasing.

Fitch-style Natural Deduction does not emphasize (as much as Gentzen-style natural deduction) the importance of the normalization procedure. Clearly one can translate from Fitch-ND proofs to Gentzen-ND proofs, normalize them and translate back into Fitch-style. But it would be interesting to see whether we could say something about the normalization procedure staying in the Fitch-style system throughout. More importantly, one can translate Fitch-ND proofs into Gentzen-ND ones and then provide categorical semantics for the Gentzen-style system, but a more direct route would be preferred. Especially because the Fitch-style does seem to accommodate more variation in the logical system.

### 4.2 Gentzen-style Modal Natural Deduction

We now present a second Natural Deduction calculus for the intuitionistic modal system $\mathsf{CK}$. This calculus can be seen as revisiting Bellin's ideas in [Bellin, 1985], except that we simplify our basis to a *constructive* one and we add terms to the natural deduction system, to obtain a Curry-Howard isomorphism[4]. We have judgements of the form

$$\Gamma \vdash M{:}\,A$$

meaning that (logically) $M$ is a proof of the Natural Deduction sequent $\Gamma \vdash A$. There are usual rules for conjunctions and implications. There is a single rule for the modality, which requires some explanation. The introduction rule for necessity, which in sequent calculus is the usual

$$\frac{\Gamma \vdash B}{\Box\Gamma \vdash \Box B}$$

[4]Note that the main goal of the paper[Bellin, 1985] was to deal with the Gödel-Löb provability logic GL, the work on $\mathsf{K}$ was only necessary background to it.

when looked at in tree-shape, requires that

$$\begin{array}{c} A_1, A_2, \ldots, A_k \\ \vdots\, \pi \\ B \end{array}$$

is transformed into a tree like

$$\dfrac{\Box A_1, \Box A_2, \ldots, \Box A_k \qquad \begin{array}{c} A_1, A_2, \ldots, A_k \\ \vdots\, \pi \\ B \end{array}}{\Box B}$$

where $A_1, \ldots, A_k$ have all been closed for substitutions. The only place in this tree where substitutions can occur, is on top of the boxed $A_i$, ie $\Box A_i$. This shape of rule, which was fairly sensible in the S4-case (as $\Box A_i$ does imply $A_i$) is less obvious here, but still reasonable. Thus the rule in Martin-Loef̈ ND looks like:

$$\dfrac{\Delta \vdash \vec{N}\colon \vec{\Box A} \qquad x_1\colon A_1, x_2\colon A_2, \ldots, x_k\colon A_k \vdash M\colon B}{\Delta \vdash \texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{x}\colon \Box B}$$

where $\vec{\Box A} = \Box A_1, \Box A_2, \ldots, \Box A_k$ and $\vec{N}$ means $N_1, \ldots, N_k$, so $\Delta \vdash \vec{N}\colon \vec{\Box A}$ really means a collections of derivations $\Delta \vdash N_1\colon \Box A_1, \ldots, \Delta \vdash N_k\colon \Box A_k$.

Note that this introduction rule for $\Box$ is mixed, it already mentions the $\Box A_i$ that we are defining how to introduce, a bad characteristic of modal Natural Deduction, which is similar to the S4 case.

This calculus is equivalent to its sequent and axiomatic formulations above and satisfies normalization, which should have been an easy corollary of work in [Bellin, 1985]).

But getting the right orientation for the equality rules given by the normalization process can be confusing. As it turns out, our original reduction rules in [Bellin *et al.*, 2001] were sloppy and needed to be tightened up. The necessary corrections were made by Kakutani[Kakutani, 2007], who proves, all the required results for the $\Box$-fragment.

THEOREM 2 (Kakutani). *Subject Reduction, Strong Normalization (SN) and the Church-Rosser property hold for the $\Box$-fragment of the calculus above.*

The system satisfies the subject reduction property, the system is strongly normalizing, the system is confluent and the system has the subformula property. Kakutani also provides call-by-name and call-by-value versions of the calculus, and a continuation passing style (CPS) transformation from the call-by-value version to the the call-by-name one, proved sound and complete. Details can be found in [Kakutani, 2007].

### 4.3 Dual-context Modal Natural Deduction

The third version of a Natural Deduction calculus for $\mathsf{CK}$ follows a pattern exploited recently by Girard (LU), Miller, Barber and Plotkin (DILL) amongst others [Ghani *et al.*, 1998]: contexts are divided into two zones, one where assumptions have modal ($\Box$) types and the other where assumptions have the usual (intuitionistic) types. We have judgements of the form

$$\Gamma|\Delta \vdash M\colon A$$

meaning that (logically) $M$ is a proof of the sequent $\Box\Gamma, \Delta \vdash A$.

DEFINITION 3. Consider the calculus DK, with the following raw terms:
$M ::= x \mid a \mid \lambda a\colon A.M \mid MM \mid \texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a} \mid \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ M$

The typing rules are the usual typing rules for the simply typed $\lambda$-calculus plus the following rules for the modality $\Box$:

$$\frac{}{\Gamma, x\colon A|\Delta \vdash x\colon \Box A}$$

$$\frac{\Gamma|\_ \vdash \vec{N}\colon \Box\vec{A} \qquad \_|\vec{a}\colon \vec{A} \vdash M\colon A}{\Gamma|\_ \vdash \texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a}\colon \Box A} \qquad \frac{\Gamma|\Delta \vdash M\colon \Box A \qquad \Gamma, x\colon A|\Delta \vdash N\colon B}{\Gamma|\Delta \vdash \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N\colon B}$$

where by $\vec{a}\colon \vec{A}$ we mean $a_1\colon A_1, a_2\colon A_2, \ldots a_n\colon A_n$.

Note that unlike our previous work on CS4 here the right-hand side of the context does not mean without a box. Actually the left-side means with at least one box, possibly more. So there might be boxed formulas on the right-hand side of the context.

We can built in all the necessary substitutions in the rule:

$$\frac{\Gamma|\_ \vdash \vec{N}\colon \Box\vec{A} \qquad \_|\vec{a}\colon \vec{A} \vdash M\colon A}{\Gamma|\_ \vdash \texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a}\colon \Box A}$$

The rule for $\Box$-elimination is a bit of a trick to get terms that look like introduction and elimination. This elimination rule can be seen as explaining how to move formulae away from the modal side of the context.

To obtain a reduction calculus we introduce the obvious reduction rules. We consider only $\beta$-rules and commuting conversions at the moment. We have two $\beta$-rules, namely:

$$\begin{array}{rcl} (\lambda a\colon A.M)N & \rightsquigarrow & M[N/a] \\ \texttt{Unbox}\ (\texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a})\ \texttt{for}\ \Box x\ \texttt{in}\ R & \rightsquigarrow & R[\texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a}/x] \end{array}$$

and the following commuting conversions:

$$(\texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N)R \quad \rightsquigarrow \quad \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ NR$$

$$\begin{array}{rl} \texttt{Unbox}\ (\texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N)\ \texttt{for}\ y\ \texttt{in}\ R & \rightsquigarrow \\ \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ \texttt{Unbox}\ N\ \texttt{for}\ y\ \texttt{in}\ R & \end{array}$$

$$\begin{array}{rl} \texttt{Box}\ M\ \texttt{with}\ \vec{N}, \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N, \vec{R}\ \texttt{for}\ \vec{b}, a, \vec{a} & \rightsquigarrow \\ \texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ \texttt{Box}\ M\ \texttt{with}\ \vec{N}, N, \vec{R}\ \texttt{for}\ \vec{b}, a, \vec{a} & \end{array}$$

While the $\beta$-conversion rules are clearly necessary, we are still working on (newer) categorical models that explain why the commuting conversions are adequate [Ritter and de Paiva, 2012].

We show subject reduction, strong normalisation and confluence for the box-fragment system $\mathsf{DK}$ discussed in this section.

PROPOSITION 4 (Subject Reduction). *Assume that term $M$ has type $A$ in the context $\Gamma|\Delta$, $\Gamma|\Delta \vdash M{:}\,A$ and that $M$ reduces to $N$ using the rules above $M \rightsquigarrow N$. Then also $N$ has type $A$, $\Gamma|\Delta \vdash N{:}\,A$.*

For strong normalisation we define a translation from our calculus into the dual calculus for $\mathsf{CS4}$, $\mathsf{DIML}$[Ghani *et al.*, 1998], which preserves reductions. This translation is given in the following definition.

DEFINITION 5. For each term $M$ define a $\mathsf{DIML}$-term $(M)^D$ by induction over the structure of $M$ as follows:

$$
\begin{array}{rcl}
(a)^D & = & a \\
(x)^D & = & \Box x \\
(\lambda a.M)^D & = & \lambda a.(M)^D \\
(MN)^D & = & (M)^D(N)^D \\
(\texttt{Box}\ M\ \texttt{with}\ \vec{N}\ \texttt{for}\ \vec{a})^D & = & \texttt{let}\ (\vec{N})^D\ \texttt{be}\ \Box\vec{x}\ \texttt{in}\ \Box(M)^D[\vec{x}/\vec{a}] \\
(\texttt{Unbox}\ M\ \texttt{for}\ \Box x\ \texttt{in}\ N)^D & = & \texttt{let}\ (M)^D\ \texttt{be}\ \Box x\ \texttt{in}\ (N)^D
\end{array}
$$

This translation preserves typing:

LEMMA 6. *Assume $\Gamma|\Delta \vdash M{:}\,A$ in the $\mathbf{IK}_\Box$ calculus. Then $\Gamma|\Delta \vdash (M)^D{:}\,A$ in $\mathsf{DIML}$.*

**Proof.** Easy induction over the definition of $(M)^D$. ■

Strong normalisation follows now directly:

THEOREM 7. *The $\mathbf{IK}_\Box$ calculus is strongly normalising.*

**Proof.** One shows that if $M \rightsquigarrow N$, then $(M)^D \rightsquigarrow^+ (N)^D$. But $\mathsf{DIML}$ is strongly normalising. Hence $\nu(M) \leq \nu((M)^D) < \infty$, where $\nu(M)$ is the length of the longest reduction sequence of $M$ if $M$ is strongly normalising and is $\infty$ otherwise. ■

Confluence cannot be inferred directly from confluence of $\mathsf{DIML}$ but needs to be shown separately.

THEOREM 8. *The $\mathbf{IK}_\Box$ calculus is confluent.*

**Proof.** As we have already shown strong normalisation it suffices to show local confluence. It is now possible to see that all critical pairs can be completed. ■

So we invented a more complicated syntax, proved the meta-theoretical results we usually need for lambda calculi for it, but we have not shown, yet, that this is really equivalent to the system $\mathsf{CK}_\Box$ that we started from. For the equivalence proof one can use a Hilbert-style presentation with the **K** axiom $\Box(A \to B) \to \Box A \to \Box B$ together with the necessitation inference rule

$$\frac{\_|\_ \vdash M\colon A}{\_|\_ \vdash \Box M\colon \Box A}$$

(i) Assume $\Gamma|\Delta \vdash M\colon A$. Then there exists a system $\mathsf{K}$ derivation of $\Box\Gamma \to \Delta \to A$.

(ii) Assume there exists a system $\mathsf{K}$ derivation of $\Gamma \to A$. Then there exists a term $M$ such that $\Gamma \vdash M\colon A$.

**Proof.**

(i) The only interesting case is the $\Box I$-rule. For this, we have the following derivation:

$$\cfrac{\cfrac{\Delta \to A}{\Box(\Delta \to A)}\,necessitation \qquad \Box(\Delta \to A) \to \Box\Delta \to \Box A}{\Box\Delta \to \Box A}\,MP$$

and now use modus ponens again.

(ii) For the other direction, necessitation is obvious from the $\Box I$-rule with empty left-hand side and empty set of terms $\vec{M}$.

Now observe for a start that we have terms $x\colon A|\_ \vdash x\colon \Box A$ and for any term $N$ in $\Gamma, x\colon A|\Delta \vdash N\colon B$ we have $\Gamma|\Delta, a\colon \Box A \vdash \mathtt{Unbox}\ a\ \mathtt{for}\ \Box x\ \mathtt{in}\ N\colon B$. Hence if we are asked to construct a term $\_|\Box\Gamma, \Delta \vdash M\colon A$ it suffices to give a term $\Gamma|\Delta \vdash M'\colon A$. Hence for the axiom it suffices to give a term $x\colon A \to B, y\colon A|\_ \vdash M\colon \Box B$. Indeed, we have

$$x\colon A \to B, y\colon A|\_ \vdash \mathtt{Unbox}\ a, b\ \mathtt{for}\ x, y\ \mathtt{in}\ Box(ab)\colon \Box B$$

■

## 5 Categorical Models

Categorical models distinguish between different proofs of the same formula. A category consists of objects, which model the propositional variables, and for every two objects $A$ and $B$ each morphism in the category from $A$ to $B$, corresponds to a proof of $B$ using $A$ as hypothesis.

*Cartesian closed categories* are the categorical models for (disjunction-free) intuitionistic propositional logic. For a careful explanation the reader should consult [Lambek and Scott, 1985]: here we just outline the intuitions. Conjunction is modelled by cartesian products, a suitable generalisation of the products in Heyting algebras. Logically the definition of a product says that a proof of a conjunction $A \wedge B$ from $C$ corresponds to a proof of $A$ (from $C$) and a proof of $B$ (from $C$). The usual logical relationship between conjunction and implication

$$A \wedge B \longrightarrow C \text{ if and only if } A \longrightarrow (B \to C)$$

is modelled by an adjunction and it defines categorically the implication connective. That is we require that for any two objects $B$ and $C$ there is an object $B \to C$ such that there is a bijection between morphisms from $A \wedge B$ to $C$ and morphisms from $A$ to $B \to C$. Disjunctions could be modelled by co-products, again a suitable generalisation of the sums of Heyting algebras, but since they cause a few problems, we prefer to restrict our language. True and false are modelled by the empty product (called a terminal object) and empty co-product (the initial object), respectively. Finally negation, as traditional in constructive logic, is modelled as implication into falsum.

Categorical models for the Gentzen-system presented $\mathsf{CK}$ are not problematic: clearly we need a cartesian closed category to model implications and conjunctions and a *monoidal* endofunctor to model the modality $\Box$. The monoidicity of the functor corresponds to the modelling of the K characteristic axiom $\Box(A \to B) \to \Box A \to \Box B$.

In previous work [Bierman and de Paiva, 2000] it was shown that to model the S4 necessity $\Box$ operator one needs a *monoidal comonad.* Here we have less structure and hence only the monoidal endofunctor remains.

DEFINITION 9. A $\mathsf{CK}_\Box$-category consists of a cartesian closed category $\mathcal{C}$, together with a monoidal endofunctor $\Box\colon \mathcal{C} \longrightarrow \mathcal{C}$. The functor being monoidal means the existence of a natural transformation with components: $m_{A,B}\colon \Box A \times \Box B \to \Box(A \times B)$ and of a morphism $m_1\colon 1 \to \Box 1$ satisfying the well-known commuting diagrams below.

The soundness theorem shows in detail how the categorical semantics models the modal logic.

THEOREM 10 (Soundness). *Let $\mathcal{C}$ be any $\mathsf{CK}_\Box$-category. Then there is a canonical interpretation $[\![\_]\!]$ of $\mathsf{CK}_\Box$ in $\mathcal{C}$ such that*

- *a formula $A$ is mapped to an object $[\![A]\!]$ of $\mathcal{C}$;*
- *a natural deduction proof $\psi$ of $B$ using formulae $A_1, \ldots, A_n$ as hypotheses is mapped to a morphism $[\![\psi]\!]$ from $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$ to $[\![B]\!]$;*
- *each two natural deduction proofs $\psi$ and $\psi$ of $B$ using formulae $A_1, \ldots, A_n$ as hypotheses which are equal (modulo normalisation of proofs) are mapped to the same morphism, in other words $[\![\phi]\!] = [\![\psi]\!]$.*

**Proof.** We use an induction over the structure of natural deduction proofs. We describe the modality rule. Consider a proof $\psi$

$$\dfrac{\begin{array}{ccc} \Gamma_1 & & \Gamma_n \\ \vdots\, \phi_1 & & \vdots\, \phi_n \\ \Box A_1 & \cdots & \Box A_n \end{array} \qquad \begin{array}{c} [A_1 \cdots A_n] \\ \vdots\, \phi \\ B \end{array}}{\Box B}\;\Box_{\mathcal{I}}$$

By induction hypothesis, let $f_1, \ldots, f_n, f$ be the interpretation of $\phi_1, \ldots, \phi_n, \phi$ respectively. Then the interpretation of $\psi$ is

$$(\Box f) \circ m_{A_1,\ldots,A_n} \circ (f_1 \times \cdots \times f_n)$$

where $m_{A_1,\ldots,A_n}$ is inductively defined by

$$m_{A_1,\ldots,A_{m-1},A_m} = m_{A_1\times\cdots\times A_{m-1},A_m} \circ (m_{A_1,\ldots,A_{m-1}}) \times id_{A_m}$$

We omit the routine verification that the desired equalities hold. ■

A trivial degenerate example of an $\mathsf{CK}_\Box$-category consists of taking any ccc, say $\mathsf{Sets}$ for example and considering the identity functor (as a monoidal functor) on it. Less trivial, but still degenerate models are Heyting algebras (the poset version of a bi-ccc) together with a closure operator.

To prove categorical completeness we use a term model construction.

THEOREM 11 (Categorical Completeness).

*(i)* *There exists a $\mathsf{CK}_\Box$-category such that all morphisms are interpretations of natural deduction proofs.*

*(ii)* *If the interpretation of two natural deduction proofs is equal in all $\mathsf{CK}_\Box$-categories, then the two proofs are equal modulo proof-normalisation in natural deduction.*

**Proof.** We show both statements by constructing an $\mathsf{CK}_\Box$-category $\mathcal{C}$ out of the natural deduction proofs. We give here only the morphisms, and omit the verification that the required equalities between proofs hold. We write a natural deduction proof

$$\frac{\begin{array}{c}A\\ \vdots\end{array}}{B}$$

as $A \vdash B$. The objects of the category are formulae, and a morphism between $A$ and $B$ is a proof of $B$ using $A$ as a hypothesis. The identity morphism is the basic axiom $A \vdash A$, and composition is given by cut. The bi-cartesian closed structure of $\mathcal{C}$ follows in the usual way from the conjunction, disjunction and implication in intuitionistic logic.

The $\Box$-modality gives rise to a monoidal functor. The functor $\Box$ sends an object $A$ to $\Box A$ and a morphism $f\colon A, B \vdash C$ to the morphism $\Box f\colon \Box A, \Box B \vdash \Box C$, obtained by applying the $\Box\mathcal{I}$-rule. Note that the monoidicity of the functor is only used to glue together the proofs of the several boxed assumptions into the original proof: if $f\colon A_1, A_2, A_3 \vdash B$, then we need

$$m_{A_1,A_2,A_3}\colon \Box A_1 \times \Box A_2 \times \Box A_3 \to \Box(A_1 \times A_2 \times A_3)$$

This category $\mathcal{C}$ shows now the claim: Assume an equation between proofs holds in all $\mathsf{CK}_\Box$-categories. Because $\mathcal{C}$ is a $\mathsf{CK}_\Box$-category, it holds in $\mathcal{C}$. But equality in $\mathcal{C}$ is equality between natural deduction proofs, hence the two proofs are equal.

■

## 6 Conclusions and more work

We have discussed several constructive versions of a basic modal system $\mathsf{K}$, including Simpson's $\mathsf{IK}$, Borghuis' Fitch-style calculus $\mathsf{K}_b$, constructive $\mathsf{CK}$ (the system inspired by Prawitz's $\mathsf{S4_I}$) and the dual context calculus $\Box$-only fragment $\mathsf{DK}$.

What is bad about this proliferation of calculi and models? For the necessity only fragment, we have almost as good a situation as for constructive $\mathsf{S4}$. We have parallel systems and we have subject reduction, normalization and confluence for the parallel lambda-calculi. We have categorical models easy to state and prove correct, but these do not constrain us much. There are many cartesian closed categories with monoidal endofunctors around, but none stands out as being particularly informative. Maybe this is right, the system is really this weak, we need to check its applications (and/or extensions) to see the usefulness of the categorical semantics. Or maybe we are missing one essential part in the modelling. In any case, it would be nice to see a more mathematically-independent model, a less syntax-derived construction.

One big positive point, that we have not described here, is that we now have a categorical model that works directly for the system $\mathsf{DK}$. This is important as the calculus $\mathsf{DK}$ is structurally similar to most of the systems produced by Pfenning and his collaborators. Until recently we could only provide categorical semantics for those systems by first translating them into single contexts calculi. These new 'fibrational models' fit well with the modelling of more expressive logics, but require more sophisticated tools, reason why they need a separate manuscript.

What is good about this proliferation of type theories and models for basic constructive modal logic? Well, the classic system $\mathsf{K}$ is still one of the most basic modal logics extant and that fact that we can do (more than one) type theory for it, and produce more than one kind of categorical model for it is very good indeed and needs more exposure. The 'choices' between calculi mostly come down to the possibility-fragment and the modelling of that lags behind. Modelling $\Diamond$s in the style of $\mathsf{CK}$ is easy, we have terms and categorical models for this system. But we do not have a developed dual contexts syntax nor do we have fibrational categorical models for it. And we do not have categorical models for Simpson-style calculi either. Given the work on intuitionistic hybrid systems [Braüner and de Paiva, 2006] this ought to be feasible, but we are not there, yet.

## BIBLIOGRAPHY

[Alechina *et al.*, 2001] Natasha Alechina, Michael Mendler, Valeria de Paiva, and Eike Ritter. Categorical and Kripke Semantics for Constructive Modal Logics. In *Computer Science Logic (CSL01), Paris*, 2001.

[Barber, 1996] A. Barber. *Dual Intuitionistic Linear Logic.* The Laboratory for Foundations of Computer Science, School of Informatics at the University of Edinburgh, ECS-LFCS-96-347, 1996.

[Bellin *et al.*, 2001] Gianluigi Bellin, Valeria De Paiva, and Eike Ritter. Extended Curry-Howard correspondence for a basic constructive modal logic. In *Proceedings of Methods for the Modalities (M4M).* CSLI, 2001.

[Bellin, 1985] G. Bellin. A system of natural deduction for GL. Theoria*, 2: 89–114*, 1985.

[Bierman and de Paiva, 2000] Gavin M. Bierman and Valeria de Paiva. On an intuitionistic modal logic. *Studia Logica*, 65(3):383–416, 2000.

[Boolos, 1993] G. Boolos. *The Logic of Provability.* Cambridge University Press, 1993.

[Borghuis, 1994] T. J. Borghuis. *Coming to Terms with Modal Logic: On the interpretation of modalities in typed $\lambda$-calculus.* PhD thesis, Technical University of Eindhoven, 1994.

[Borghuis, 1998] Tijn Borghuis. Modal pure type systems. *Journal of Logic, Language and Information*, 7(3):265-296, 1998.

[Braüner and de Paiva, 2006] T. Braüner and V. de Paiva. Intuitionistic hybrid logic. *Journal of Applied Logic*, 4(3):231–255, 2006.

[Cirstea *et al.*, 2007] Corina Cirstea, Alexander Kurz, Dirk Pattinson, Lutz Schroeder, and Yde Venema. Modal Logics are Coalgebraic. *Electronic Workshops in Computing, The British Computer Society*, 2007.

[de Paiva and Pientka, 2011] Valeria de Paiva and Brigitte Pientka. Intuitionistic Modal Logic and Applications. *Information and Computation*, Volume 209, Issue 12:1435–1538, 2011.

[Fairtlough *et al.*, 2001] M. Fairtlough, M. Mendler, and E. Moggi. Modalities in Type Theory. *Mathematical Structures in Computer Science*, 11(4), 2001.

[Fitch, 1952] F. Fitch. *Symbolic Logic: An Introduction.* The Ronald press Co., New York, 1952.

[Gentzen, 1969] G. Gentzen. On the relation between intuitionist and classical arithmetic (1933). In M.E. Szabo, editor, *The collected papers of Gerhard Gentzen*, pages 53–67, Amsterdam, 1969. North-Holland.

[Ghani *et al.*, 1998] N. Ghani, V. Paiva, and E. Ritter. Explicit substitutions for constructive necessity. *Automata, Languages and Programming*, page 743, 1998.

[Gödel, 1986] K. Gödel. An interpretation of the intuitionistic propositional calculus (1933f). In S. Feferman, J.W. Dawson Jr., S.C. Kleene, G.H. Moore, R.M. Solovay, and J. van Heijenoort, editors, *K. Gödel's Collected Works: Volume I: Publications 1929-1936*, pages 301–302, Oxford, 1986. Oxford University Press.

[Goré *et al.*, 2004] Rajeev Goré, Valeria de Paiva, and Michael Mendler. Intuitionistic Modal Logic and Application. *Journal of Logic and Computation*, 14(4), 2004.

[Kakutani, 2007] Y. Kakutani. Calculi for intuitionistic normal modal logic. In *Proceedings of Programming and Programming Languages*, pages 234–248, 2007.

[Kripke, 1963] S. Kripke. Semantical Analysis of Modal Logic I. Normal Proposicional Calculi. *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.

[Kripke, 1965] S. Kripke. Semantical Analysis of Modal Logic II. Non-Normal Modal Propositional Calculi. In J. W. Addison, L. Henkin, and A. Tarski, editors, *The Theory of Models (Proceedings of the 1963 International Symposium at Berkeley)*, pages 206–220, Amsterdam, 1965. North-Holland.

[Lambek and Scott, 1985] J. Lambek and Ph. J. Scott. *Introduction to Higher-Order Categorical Logic.* Cambridge University Press, 1985.

[Lemmon, 1957] E. J Lemmon. New Foundations for Lewis Modal Systems. *The Journal of Symbolic Logic*, 22(2):176–186, 1957.

[Lewis and Langford, 1932] C. I. Lewis and C. H. Langford. *Symbolic Logic.* Century, 1932.

[Prawitz, 1965] D. Prawitz. *Natural Deduction: A Proof-Theoretic Study.* Almqvist and Wiksell, 1965.

[Ritter and de Paiva, 2012] Eike Ritter and Valeria de Paiva. *Fibrational Models of Modal Logic.* University of Birmingham, UK, 2012.

[Simpson, 1994] A.K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic.* PhD thesis, University of Edinburgh, 1994.

[Wijesekera, 1990] D. Wijesekera. Constructive modal logic I. *Annals of Pure and Applied Logic*, 50:271–301, 1990.