Bounded Dialectica Interpretation: categorically

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Abstract Recently Ferreira and Oliva introduced the Bounded Functional Interpretation (BFI) [7] as a way of continuing Kohlenbach's programme of shifting attention from the obtaining of precise witnesses to the obtaining of *bounds* for these witnesses, when proof mining. One of the main advantages of working with bounds, as opposed to witnesses, is that the non-computable mathematical objects whose existence is claimed by various ineffective principles can sometimes be bounded by computable ones. In this note we present first steps towards a categorical version of BFI, along the lines of de Paiva's version of Gödel's Dialectica interpretation, the Dialectica Categories, in [3]. The previous categorical constructions seem to extend smoothly to the new ordered setting.

This paper is dedicated to Luiz Carlos Pereira on his birthday

1 Introduction

This preliminary note tries to make good in the promise I have made over the years to Luiz Carlos Pereira to connect the categorical constructions in the Dialectica Categories of my thesis [3] to the actual proof theory that inspired them, that is, to the functional interpretations themselves, mostly Gödel's Dialectica interpretation, but also its Diller and Nahm variant.

This note is much indebted to the careful work done by Paulo Oliva and collaborators, especially Gilda Ferreira [8], as I hope it is abundantly clear from the note itself. But, not surprisingly, I disagree with some of their conclusions, hence the need to write this. Lastly, a word of caution: it is possible that all that I have to say here is better said (and has been said) in the higher-order (in the sense of category theory) language of [10] and [12]. But there is still a point in writing this, as the "translation" from the fibrations and higher-order category theory language to the

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pedestrian category theory used here is non trivial and many will not be able to read the very abstract formulation.

2 The Dialectica constructions

For my thesis I was originally trying to provide an internal categorical model of Gödel's Dialectica Interpretation, which I presumed would be a cartesian closed category. But the categories I came up with proved also to be models of Linear Logic. This was a surprise and somewhat of a boost for Linear Logic, which was only beginning to appear then.

The traditional categorical modeling of intuitionistic logic goes as follows: a formula A is mapped to an object A of an appropriate category, the conjunction $A \wedge B$ is mapped to the cartesian product $A \times B$ and the implication $A \to B$ is mapped to the space of functions B^A (the set of functions from A to B). These are real cartesian products, so we have projections ($A \times B \to A$ and $A \times B \to B$) and diagonals ($A \to A \times A$), which correspond to deletion and duplication of resources. This is not a linear structure. To model a linear logical system faithfully we need to use *tensor products* and *internal homs* in Category Theory. Luckily these structures were considered by category theorists long before Linear Logic, so they were easy. What was hard was to define the "make-everything-usual" operator, the modality !, which applied to a linear proposition makes it an intuitionistic one.

Definition 1. The category $\mathsf{Dial}_2(\mathbf{Sets})$ has as objects triples $A = (U, X, \alpha)$, where U, X are sets and α is an ordinary relation between U and X. (so either u and x are α related, $\alpha(u,x) = 1$ or not.)

A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of functions (f, F), where $f\colon U \to V$ and $F\colon X \to Y$ such that

$$\begin{array}{ccc} U & \xleftarrow{\ \alpha\ } & X \\ {\scriptstyle f}\downarrow & \Downarrow & \uparrow{\scriptstyle F} \\ V & \xleftarrow[\ \beta\ ]{} & Y \end{array} \qquad \forall u \in U, \forall y \in Y \quad \alpha(u, Fy) \text{ implies } \beta(fu, y)$$

$$\text{or } \alpha(u, F(y)) \leq \beta(f(u), y)$$

An object A is not symmetric: think of (U, X, α) as $\exists u.\forall x.\alpha(u,x)$, a proposition in the image of the Dialectica interpretation. But using this category and its linear, multiplicative structure we can prove:

Theorem 1 (Model of CLL modality-free, 1988). *The category* $\mathsf{Dial}_2(\mathbf{Sets})$ *has products, coproducts, tensor products, a par connective, units for the four monoidal* *structures and a linear function space, satisfying the appropriate categorical properties. The category* $\mathsf{Dia}_2(\mathbf{Sets})$ *is symmetric monoidal closed with an involution* $(\ )^*$ *that makes it a model of classical linear logic, without exponentials.*

But how do we get modalities (or as Girard calls them, exponentials)? For this specific categorical model, we need to define two special co-monads and do plenty of work using distributive laws to prove the desired theorem. Recall that $!A$ must satisfy $!A \to !A \otimes !A$, $!A \otimes B \to I \otimes B$, $!A \to A$ and $!A \to !!A$, together with several equations relating them. The difficulty is to define a comonad such that its coalgebras are commutative comonoids and such that the coalgebra and the comonoid structure interact nicely.

Theorem 2 (Model of Classical Linear Logic with modalities, 1988). *We can define comonads* T *and* S *on* $\mathsf{Dial}_2(\mathbf{Sets})$ *such that the Kleisli category of their composite* $! = T;S$, $\mathsf{Dial}_2(\mathbf{Sets})^!$ *is cartesian closed.*

Define T by saying $A = (U,X,\alpha)$ goes to (U,X^*,α^*) where X^* is the free commutative monoid on X and α^* is the multiset version of α.

Define S by saying $A = (U,X,\alpha)$ goes to (U,X^U,α^U) where X^U is the set of functions from U into X. Then compose T and S to get $A = (U,X,\alpha)$ goes to $(U,(X^*)^U,\alpha{*}^U)$. This composite comonad does get us from the linear category to the cartesian closed category corresponding to the Diller-Nahm interpretation. This was the ultimate result desired in the thesis. But on the way to proving this we had a second main definition

Definition 2. The category $\mathsf{DDial}_2(\mathbf{Sets})$ has as objects triples $A = (U,X,\alpha)$, where U,X are sets and α is an ordinary relation between U and X.

A map from $A = (U,X,\alpha)$ to $B = (V,Y,\beta)$ is a pair of functions (f,F), where $f\colon U \to V$ and $F\colon U \times X \to Y$ such that

$$\begin{array}{ccc} U & \xleftarrow{\ \alpha\ } & X \\ {\scriptstyle f}\downarrow & \Downarrow & \uparrow{\scriptstyle F} \\ V & \xleftarrow{\ \beta\ } & Y \end{array} \qquad \forall u \in U, \forall y \in Y \quad \alpha(u,F(u,y)) \text{ implies } \beta(fu,y)$$

$$\text{or } \alpha(u,F(u,y)) \leq \beta(f(u),y)$$

This category seems extremely similar to our first definition, in particular the objects of the two categories are exactly the same. But morphisms are what make a category and morphisms in the two cases are different enough, and induce very different structures in the categories. In particular for this category we cannot define a multiplicative disjunction, unlike the previous one, and the tensor product is much simpler.

Theorem 3 (Model of ILL modality-free, 1987). *The category* $\mathsf{DDial}_2(\mathbf{Sets})$ *has products, (weak)-coproducts, tensor products, units for the two monoidal structures* *and a linear function space, satisfying the appropriate categorical properties. The category* $\mathsf{DDial}_2(\mathbf{Sets})$ *is symmetric monoidal closed with products and weak co-products that makes it a model of* **intuitionistic** *linear logic, without exponentials.*

This model of restricted, intuitionisitic only linear logic has nonetheless a very special property: it has *co-free* comonoids and hence it's a rather special model of linear logic, which accounts for the theorem below

Theorem 4 (Model of Intuitonistic Linear Logic with modalities, 1987). *We can define a co-free comonad* $!_{DN}$ *on* $\mathsf{DDial}_2(\mathbf{Sets})$ *such that its co-Kleisli category* $\mathsf{Dial}_2(\mathbf{Sets})^!$ *is cartesian closed.*

Many calculations are needed to prove that the linear logic modality ! is well-defined and to obtain a model of *classical* linear logic. But my previous work did not emphasize this two-step process, as the goal was to obtain a cartesian closed category, a model of intuitionistic linear logic. The two step process goes from Dial to DDial to DNDial, the Diller-Nahm category, which is now Cartesian Closed. Here we want to see this two-step process as a reflexion of what is happening in the proof theory. The point being that going from the linear logic category Dial to the intuitonistic linear logic category DDial, which has the same objects, but different morphisms and hence different function spaces and different tensors can be seen as the result of applying the modified realizability translation, in Oliva's terms. These we recap briefly below.

3 Unification of Proof Interpretations

In a recent preprint [17] Oliva provides a summary of his proposed syntactic unification of functional interpretations, a long standing research program that started with the unification of Gödel's Dialectica Interpretation and Kreisel's Modified Realizability in 2006 [15]. This unification is purely *syntactic*, so formulae of logical systems are mapped to other logical systems, but proofs of propositions do not have to be mapped, a priori.

In principle a semantic version of this unification already existed for the Dialectica interpretation and its Diller-Nahm variant, as result of the work in [3]. The basis of Oliva's more comprehensive syntactic unification (which includes besides Diller-Nahm and Dialectica, modified realizability and their "truth"-variants, five functional interpretations so far) is the rephrasing of modified realizability in terms of *relational realizability* which talks about a relation between potential witnesses and challenges, in a way very similar to the way the dialectica interpretation is usually presented in terms of witnesses and counter-examples.

A second main idea of this unification is described in [16, 8] where it is proposed that the main characteristics of the functional interpretations can be gotten via interpretation of intuitionistic linear logic, complemented by several different modalities, in particular modalities for Gödel's dialectica, for the Diller-Nahm variant of dialectica and for modified realizability.

While Oliva's program acknowledges previous work of the present author [6, 2] as inspiration, the exact categorical relationships are not clearly spelled out. In particular the categorical modelling of Kreisel's modified realizability, in terms of realizability toposes and the work developed in a series of papers by van Oosten, Hyland and others is not, as yet, related to this unification. The goal of this note is to see where are the problems in making this connection and whether we can see a way of solving them. We start by reviewing the goals and motivations of both realizability and functional interpretations.

3.1 Realizability

Kleene [13] recounts how his idea for a numerical realizability developed as he wished to give some precise meaning to the intuition that there should be a connection between Intuitionism and the theory of recursive functions. Both theories stress the importance of extracting information effectively. Kleene starts by conjecturing a weak form of Church's Rule: if a closed formula of the form $\exists x \forall y \phi(x,y)$ is provable in intuitionistic number theory, then there must be a general recursive function F such that for all n, the formula $\phi(n, F(n))$ is true.

The main motivation for Kleene for inventing realizability in the forties seems to be the work of Hilbert and Bernays, in their Grundlagen der Mathematik. They explained the finitist position in Mathematics, as follows (translation by Jaap van Oosten in [19])

> An existential statement about numbers, i.e. a statement of the form 'there exists a number n with property $A(n)$' is finitistically taken as a 'partial judgement' that is, as an incomplete rendering of a more precisely determined proposition which consists in either giving directly a number n with the property $A(n)$ or a procedure by which such a number can be found...

Kleene wondered whether a completion procedure might be provided that completed the description for all logical connectives, but suggested that his was, in any case, a "partial analysis of the intuitionistic meaning of the statements".

Realizability is famously understood in categorical terms using Hyland's "Effective topos" [11]. Modified realizability, as originally defined by Kreisel, has also given rise to a categorical formulation in terms of toposes, so one would expect connections between these toposes and the Oliva's program of functional interpretations unification, at the semantic level.

3.2 Proof Mining and Functional Interpretations

Proof mining is the process of logically analyzing proofs in mathematics with the aim of obtaining new information. How do we do proof mining and what do we get? A proof of a theorem like "x as an element of a space X is a root of a function $f : X \to$

R" is a complete theorem, i.e. it gives us an equation $f(x) = 0$ and we don't need any other information. But a theorem stating that "f is (strictly) positive at a point $x \in X$" is incomplete, for it leaves open how far from zero the value $f(x)$ actually is. It turns out that in many cases the information missing in an incomplete theorem can be extracted by purely logical analysis out of prima-facie ineffective proofs of the theorem. The *Dialectica interpretation* together with other proof interpretations is a major tool for proof mining. But what are proof interpretations or functional interpretations? They are proof transformations that can be used to extract extra information from proofs. Ulrich Kohlenbach has made extensive use of these and explains a full range of proof interpretations and their applications in [14].

4 Semantic Unification of Proof Interpretations?

Most of the recent work on semantic proof interpretations has been using fibrations, e.g. Hyland [12], Biering [1], Hofstra [10] and Hedges [9]. While the more expressive framework might be necessary, the syntactic unification described by Oliva and collaborators gives hope that another, more pedestrian route might be available. And if the pedestrian route isn't available it will be enlightening to see where it fails vis-à-vis the syntactic work.

Taking a leaf from Shirahata [18] and Oliva and Ferreira's work we should discuss the proof interpretation of a weaker system first. While my original work was mostly concerned with intuitionistic logic, and considered Linear Logic as a step stone to get to "traditional" logic, some of the work, starting with Shirahata is about interpretations of *linear* logical systems. Some are about classical linear logic, some about intuitionistic linear or affine logic.

4.1 Intuitionistic Linear Logic

We revisit Ferreira and Oliva's "Functional Interpretation of Intuitionistic Linear Logic" [8], where we disagree, somewhat, with their interpretation of the connection between our works. They say and we agree that

> In this section we try to explain and make more explicit the link between our framework for unifying interpretations of IL via interpretations of ILL and the categorical approach on [3, 2, 6] for modelling ILL.

But while it is true that "in [2] one finds a categorical version of the Dialectica interpretation and an endofunctor interpretation for the modality !A that corresponds to the Diller-Nahm interpretation" the table setting up the correspondence in their paper is not as precise as it needs to be. In particular, what is missing from the explanation in section 5 of [8] is the two-step process that takes us from modality-free Linear Logic to Intuitionistic Logic. The first of these steps sends us from a pure (no modalities) linear category to one that might correspond to Modified Realizability in their terms, while the second step takes us from modified realizability objects to Diller-Nahm sets of witnesses/challenges.

Table 1 Comparison

	Fer-Oliva	DDial	Dial
realizers	finite types	ccc C	ccc C
formulas	$\|A\| \subseteq U \times X$	$\alpha \subseteq U \times X$	$\alpha \subseteq U \times X$
sequents	$A \vdash B$	$(f : U \to V, F : U \times Y \to X)$	$(f : U \to V, F : Y \to X)$
linear impl	$A \multimap B$	$(V^U \times X^{U\times Y}, U \times Y, \alpha \multimap \beta)$	$(V^U \times X^Y, U \times Y, \alpha \multimap \beta)$
tensor	$\otimes$	$(U \times V, X \times Y)$	$(U \times V, X^V \times Y^U)$
modality $!_R$	$!\forall x\|A\|^u_x$		(U, X^U, α^U)
modality $!_{DN}$	$!\forall x \in a\|A\|^u_x$	(U, X^*, α^*)	
modality $!_{R;DN}$	$!\|A\|^a_x$		$(U, (X^U)^*, (\alpha^U)^*)$

Note that the realisers of the functional interpretation are taken from a given (fixed) cartesian closed category C in [3], while Ferreira and Oliva work with the particular cartesian closed category of the functionals of finite type. Apart from the relation $\alpha \oplus \beta$, the interpretation of the linear logic connectives in DDial coincides precisely with the definitions in [8].

Note that [8] assumes that each finite type is inhabited by at least one element, while [3] imposes no similar restriction. (this means that Ferreira and Oliva are dealing with affine logic, instead of linear logic, as the tensor operator can now have projections given by the inhabitants of the sets.) In the category DDial we have no way of producing projection functions $\pi_{1,2}: A \times B \to A, B$, so we're dealing with linear logic, not affine logic.

While we do explain how one could, in principle 'cheat' and provide an interpretation for the contraction axiom $A \multimap A \otimes A$ by using a trick to map into one counterexample, when we have a choice of two such, this is not a uniform operation, hence it does not produce a functor, is poor category theory and we do not pursue it in our work.

Another difference is the weak coproduct of [3], which can be made much simpler in Ferreira and Oliva's setting using inhabitedness of witnesses and counterexamples. In the categorical approach we do not assume inhabitedness, but can still obtain the weak coproduct.

4.2 Modified Realizability Modality

If one thinks of modified realizability witnesses as actual and potential witnesses, as in Oliva's reformulation, where actual witnesses are a subset of the potential ones, then objects of the form (U, X^U, α^U) make sense, as we can think of U as being naturally embedded into X^U via the "constant" map $X \to X^U$. Then we might end up with a construction as follows:

Definition 3. Objects of the category MR (for modified realizability) are triples $A = (U, X^U, \alpha^U)$, $B = (V, Y^V, \beta^V)$ where the relation $\alpha^U : U \times X^U \to 2$ is the composition of $U \times X^U \to U \times X \to^{\alpha} 2$. Morphisms need to be considered in the category of coalgebras of the comonad R that sends an object A to $RA = (U, X^U, \alpha^U)$

Ferreira and Oliva state, when discussing the bang interpretation that "Our conditions are more general, and include as particular case the instance where "!" is a comonad with comonoid objects." It is true that their conditions are more general, but they are simply of the form "this needs to happen", while the categorical model exhibits a mathematical structure that indeed satisfies the requirements of the model.

4.3 Diller-Nahm Modality

If we start from the category Dial we can simply apply the Diller-Nahm modality taking (U, X, α) to (U, X^*, α^*) where we are considering not simply the free monoid in X, but actually the free *commutative* monoid on X. This corresponds precisely to the idea that for the Diller-Nahm interpretation we collect the witnesses into a finite, but unlimited set.

4.4 The composite $!_{R,DN}$ Modality

Only when we compose the two comonads we can get that objects of the form $!A$ satisfy all the conditions for modelling propositions of intuitionistic logic.

5 Dialectica Spaces over Partial Orders

Let us consider the category Poset of partially ordered sets and monotone functions. Monotone functions compose to give monotone functions and the identity function on a poset $(X, \leq)$ is monotonic. The category Poset has products. Given $(X, \leq_X)$ and $(Y, \leq_Y)$ their product is $(X \times Y, \leq_{X \times Y})$, where the order on the product is pointwise order $(x, y) \leq_{X \times Y} (x', y')$ iff $x \leq_X x'$ and $y \leq_Y y'$.

The category Poset is cartesian closed, the function space of Poset is given by monotone maps ordered pointwise $f \leq g : U \to V$ iff $f(u) \leq_V g(u)$ for all u in U. The category Poset also has coproducts, given by the disjoint sum.

What can we say about the dialectica constructions over the category Poset? In previous work we described two Dialectica-constructions: they share the objects, but morphisms in DDial are more complicated than the ones in Dial. The category Dial was originally called *GC* as a thank-you note to Girard, who originally suggested it as a simplification of the dialectica DDial construction.

First we can define Dial$_2$Posets.

Definition 4. Objects of Dial$_2$Posets are triples $A = (U,X,\alpha)$ where $(U,\leq_U)$ and $(X,\leq_X)$ are posets and $\alpha{:}U \times X \to 2$ is a generalized relation into 2. Maps are monotone maps $f{:}U \to V$ and $F{:}Y \to X$ such that $\alpha(u,F(y)) \leq \beta(f(u),y)$. This means that in the following commuting diagram we have a 2-cell:

$$\begin{array}{ccc} U \times Y & \xrightarrow{U \times F} & U \times X \\ {\scriptstyle f \times Y}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ V \times Y & \xrightarrow[\beta]{} & 2 \end{array}$$

The identity map on an object $A = (U,X,\alpha)$ consists of the identity on U and the identity in X, (id_U, id_X) which clearly satisfies the implication condition.

Composition of morphisms is straightforward. Given morphisms $(f,F){:}A \to B$, where B is (V,Y,β) and $(g,G){:}B \to C$, where C is (W,Z,γ), the composition in the first coordinate is simply $f;g{:}U \to W$ and in the second coordinate we have $Z \xrightarrow{G} V \xrightarrow{F} X$. We are always composing monotone maps, which gives us monotone maps and we need to check that $\alpha(u,F(G(z))) \leq \gamma(g(f(u)),z)$. But because (f,F) is a morphism we know $\alpha(u,F(G(z))) \leq \beta(f(u),G(z))$ and because (g,G) is a morphism we know $\beta(f(u),G(z)) \leq \gamma(g(f(u),z)$, so putting the inequalities together we have the desired one.

As discussed in [4] we need proper products and function spaces in Posets to define tensor products and function spaces in Dial$_2$Posets. Since the category Posets does have products and function spaces we can define

Definition 5. Given objects A and B, say (U,X,α) and (V,Y,β), of Dial$_2$Posets their function space $A \multimap B$ is the object $(V^U \times X^Y, U \times Y, \alpha \multimap \beta)$ where the relation $\alpha \multimap \beta$ is defined by $\alpha \multimap \beta([h,H],[u,y])$ iff whenever $\alpha(u,H(y))$ holds then $\beta(h(u),y)$ holds.

This is a direct internalization of the notion of dialectic morphism above and reverse engineering from this notion of morphism (to produce a monoidal closed category) gives us the following notion of tensor product.

Definition 6. Given objects A and B of Dial$_2$Posets ((U,X,α) and (V,Y,β) respectively), their tensor product $A \otimes B$ is the object $(U \times V, X^V \times Y^U, \alpha \otimes \beta)$ where the relation $\alpha \otimes \beta$ is defined by $\alpha \otimes \beta([u,v],[h_1,h_2])$ iff whenever $\alpha(u,h_1(v))$ and $\beta(h(v,h_2(v))$ hold.

Then, all going according to plan, we end up with a theorem that says:

Theorem 5. *The category* $\mathsf{Dial_2Posets}$ *is a symmetric monoidal closed category, with products and coproducts.*

Products are given by products in the first coordinate and coproducts in the second, so if $A = (U,X,\alpha)$ and $B = (V,Y,\beta)$ then the product $A \times B = (U \times V, X+Y, p)$ where $p: U \times V \times (X+Y) \to 2$ chooses either α or β depending on the element of $X+Y$ picked. A terminal object will be $1 = (1,0,\iota)$ where ι is the empty relation on 1×0.

Natural Numbers Objects

The category of sets has the original natural numbers object N, with zero and successor $\mathsf{zero}, \mathsf{suc}$ functions. One might wonder about a natural numbers object in $\mathsf{Dial_2Sets}$. By analogy one would expect it to be something like (N,X,α), where N is the natural numbers in sets and X would be some kind of dual of natural numbers. While a dual of the natural numbers is hard to conceive in sets, it would make sense to simply invert the order, if considering the concept in Poset.

Say we had such a natural numbers object $N = (N,N,\alpha)$ in $\mathsf{D_2Posets}$, where $\alpha: N \times N \to 2$ is the diagonal. As discussed in [5], were we to consider a NNO structure with respect to the categorical product in $\mathsf{Dial_2Sets}$, we would need a map zero $z: 1 \to N$. Since the terminal object in $\mathsf{Dial_2Sets}$ is $(1,0,ch)$, the empty relation in the empty set, this would be $(z: 1 \to N, \bar{z}: N \to 0)$ and there is no map $\bar{z}: N \to 0$. But degenerate linear NNO's, using the monoidal structure of the tensor product in $\mathsf{Dial_2Sets}$ can be obtained as suggested in the note above.

6 Conclusions

We have re-appraised the work on Dialectica constructions under the light of the unification of functional interpretations carried out by Gilda Ferreira and Paulo Oliva [8]. The main result is our improved table of comparisons between our works.

Further we described an easy extension of the formalism of Dialectica constructions to ordered categories, which we claim should model the bounds on witnesses and counter-examples as described in the Bounded Functional Interpretation of Fernando Ferreira and Paulo Oliva [7]. We have not solved the main problem we wished to investigate, that is the connection between the Effective Topos (and the Modified Realizability topos) and the categorical working arising from Dialectica and its connection to Linear Logic. But we have already 'borrowed' more time than we can afford, to finish this note. It is a small souvenir (*uma lembrancinha*) for Luiz Carlos and it will have to do.

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