Dialectica Categories, Cardinalities of the Continuum and Combinatorics of Ideals

SAMUEL G. DA SILVA, *Departamento de Matemática, Instituto de Matemática, Universidade Federal da Bahia, Campus Ondina, Av. Adhemar de Barros s/n, 40170-110, Salvador/Bahia, Brazil. E-mail: samuel@ufba.br*

VALERIA C. V. DE PAIVA, *Nuance Communications, Sunnyvale, CA 94085, USA. E-mail: valeria.depaiva@gmail.com*

Abstract

Andreas Blass has frequently pointed out that inequalities between *cardinal invariants of the continuum* are usually proved via morphisms of some versions of (dual) Dialectica Categories – which are certain categories introduced by the second author as categorical models of linear logic. In this paper, we discuss the reasons why Dialectica Categories can be successfully applied to prove such inequalities. The main goal of this ongoing research is to circumscribe the effectivity of the described method and to discover why it works as well as it does. Combinatorics of ideals and the notions of unboundedness and domination in pre-orders are presented as study cases which serve as evidence in favour of some conjectural principles. To finish, a number of questions and problems are posed.

Keywords: Cardinal Invariants of the Continuum, Category Theory, Dialectica Categories.
MSC Primary 03E17, 18A15; Secondary 18A05, 18A20.

1 Introduction

Set Theory and Category Theory will be certainly present in any list of natural candidates to provide foundations for all Mathematics. However, as opposite players in a never ending game, researchers from both areas often decide not to consider arguments, reasoning and techniques from the "other side" in their own work. In this paper, the authors (one of them a set-theorist, the other a category theorist) reject such way of doing Mathematics. Indeed, this work investigates a certain context where both Category Theory and Set Theory interact – in a splendid way –, and, in this particular context, we want to explain certain *general phenomena*.

> . . . It often happens that there are similarities between the solutions to problems, or between the structures that are thrown up as part of the solutions. Sometimes, these similarities point to more general phenomena that simultaneously explain several different pieces of mathematics. These more general phenomena can be very difficult to discover, but when they are discovered, they have a very important simplifying and organizing role, and can lead to the solutions of further problems, or raise new and fascinating questions.

(T. Gowers, *The Importance of Mathematics*, 2000 [14])

Given the expected audience for this work – logicians with background, experience and/or interests in Category Theory –, we assume the reader is familiar with the usual language of categories; in any case, we refer to [18] or [1] for more details on that. Certain technicalities of Set Theory are needed in this paper (probably with some level of specificity which is higher than the expected for researchers who do not work with such notions in a daily basis), so let us talk a little bit about sets.

If x and y are sets, then $x \times y$, ${}^{x}y$ and $\mathcal{P}(x)$ denote, respectively, the Cartesian product of x and y, the set of functions from x into y and the *powerset* – i.e., the set of all subsets – of x. Two sets x, y are said to be *equipotent* (and we write $x \approx y$) if there is a bijection between them – and, intuitively, they have the same size.

The set theoretical framework of this paper is the usual **ZFC** (*Zermelo-Fraenkel – **ZF**, with Choice*) Set Theory – with first order, finitary subjacent logic –, which will be freely used; our terminology and notations are standard, see for instance [17]. In particular, $\alpha, \beta, \gamma, \delta$ denote ordinals and κ, λ, θ denote cardinals. Recall that (*i*) ordinals are transitive sets which are well-ordered by $\in$; (*ii*) the family of all ordinals, **On**, is not a set – it is a *proper class*; and (*iii*) for every α one has $\alpha = \{\beta : \beta < \alpha\}$ – i.e., an ordinal is precisely the set of all ordinals which $\in$-precede it.

We are assuming that the Axiom of Choice holds, so it follows that every set can be well-ordered – and thus we can define the *cardinality* of any set x (denoted by $|x|$) as the minimal α such that $\alpha \approx x$. An ordinal κ is said to be *cardinal* if $|\kappa| = \kappa$. In this sense, cardinals are *initial ordinals*. Clearly, $x \approx y$ if, and only if, $|x| = |y|$.

The *successor* of a cardinal κ – which is the smallest cardinal greater than κ – is denoted κ^+ and one can easily check (using Replacement) that $\kappa^+ = \{\alpha : \alpha \preccurlyeq \kappa\}$ – where $\preccurlyeq$ is the usual *domination relation*, meaning that $x \preccurlyeq y$ indicates that there is a (nameless) injective function from x into y. In particular, the first uncountable ordinal, ω_1, is precisely the set of all countable ordinals.

Even without the Axiom of Choice one can prove that if x and y are sets such that $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x \approx y$ – indeed, this is the famous Schröder-Bernstein-Cantor Theorem, which holds in **ZF**.

In a Set Theory setting as the one described above (where choice is fully embraced), all cardinals of infinite sets are *alephs* – meaning that, considering the sequence (indexed by all ordinals and defined by transfinite recursion) given by

$$\aleph_0 = \omega = \{n : n \text{ is a natural number}\},$$

$$\aleph_{\alpha+1} = \omega_{\alpha+1} = (\aleph_\alpha)^+$$

and, if γ is a limit ordinal,

$$\aleph_\gamma = \omega_\gamma = \sup\{\aleph_\alpha : \alpha < \gamma\},$$

then the class of all infinite cardinalities **Card** is given by $\mathbf{Card} = \{\aleph_\alpha : \alpha \in \mathbf{On}\}$.

The *Continuum Hypothesis* (denoted by **CH**) is the statement "$\aleph_1 = \mathfrak{c}$", where $\mathfrak{c}$ denotes the cardinality of the *continuum*, that is, $\mathfrak{c} = 2^{\aleph_0} = |\mathbb{R}| = |{}^{\omega}2| = |{}^{\omega}\omega|$. The Continuum Hypothesis is equivalent (in **ZFC**) to the following statement: "If X is an infinite subset of $\mathbb{R}$, then either X is countable (i.e., $X \approx \omega$) or $X \approx \mathbb{R}$" – indeed, this is the original statement of **CH**, due to Cantor (1879). Recall that Cantor also proved (using the *diagonal argument*) that $\kappa^+ \leqslant 2^\kappa$ for every κ; the *Generalized Continuum Hypothesis* (**GCH**) asserts equality everywhere, that is, "$2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every α".

After this quick recap, let us describe the organization of this paper. In Section 2 we recall some facts on Dialectica Categories, which were introduced by the second author in [23]. We also present a version of (the dual) of this category, which was, independently, introduced by Vojtáš ([30]). Applications in Set Theory and Analysis of Vojtáš' category are discussed. In Section 3 we investigate the so-called *cardinal invariants of the continuum*, and we show that several of these cardinals may be associated to objects of the categories considered. In Section 4 we discuss a certain *empirical fact*, which was first pointed out by Blass, and we present some conjectural principles, which we see as steps towards an explanation of this empirical fact; *combinatorics of ideals* shows up as a crucial study case – as do the notions of unboundedness and domination in *pre-orders* (i.e., reflexive transitive relations). We conclude by posing some questions and presenting some directions for the upcoming research.

Before getting into the paper itself, the authors would like to present a disclaimer: this work is a report of ongoing research. There are few new results in this paper (only the Theorems 4.7 and 4.13 appear here for the first time). The authors believe this paper is worth sharing because of the novelty of its study cases, conjectures and problems; this is a new perspective on some traditional material. The paper, which summarizes and surveys a number of known results in order to give context to its conjectures and problems, is not supposed to be our final word on this subject; we do not have yet, for instance, the answer to our Main Question (concerning "Blass' empirical fact" , see Section 4). This is a first step on a path that was never taken before (which is to *explain* why a certain method works as well as it does); but any path needs a first step, and this is it.

2 Dialectica Categories

2.1 Some history

Dialectica categories are one of the outcomes of a long line of research in Categorical Logic, and their history can be traced back to Gödel. Gödel developed a functional interpretation of intuitionistic logic via computable (or primitive recursive) functionals of finite type, using the so-called *Dialectica Interpretation* – named after the Swiss journal Dialectica, special volume dedicated to Paul Bernays 70th birthday in 1958, see [13] –, in order to provide a proof of the consistency of arithmetic.

Almost three decades after that, Hyland (second author's PhD advisor) suggested that to provide a categorical model of the Dialectica Interpretation, one should look at the functionals corresponding to the interpretation of logical implication. The categories de Paiva came up with in her thesis (see [23] and [24]) proved to be a model of Linear Logic. Linear Logic had then just been introduced by Girard (see [12]) as a proof-theoretic tool, combining the dualities of classical logic and the constructive content of proofs of intuitionistic logic. Since then, Linear Logic has been recognized as an important tool for the semantics of Computing.

It is interesting to notice that the motivations for considering the Dialectica interpretation were quite varied as time went by. For Gödel (in 1958), the interpretation was a way of proving consistency of arithmetic. For de Paiva (in 1988), it was an internal way of modelling the Dialectica interpretation that turned out to produce models of Linear Logic instead of models of Intuitionistic Logic, the expected ones. For Blass (in 1995) [3] Dialectica Categories were used as a way of connecting the work of Vojtáš' on Set Theory to de Paiva's and also to his own work on Linear Logic and to *cardinal invariants of the continuum* (which are the "cardinalities of the continuum" we refer to in the title of this paper).

The second author of this paper did not foresee that applications of Dialectica categories would arise in Set Theory at all. In fact, the present work emerged from the desire to provide an explanation of why such applications exist and why they work as well as they do.

2.2 Dialectica Categories: Basic Definitions

Objects of the Dialectica category $\mathrm{Dial}_2(\mathbf{Sets})$ are triples, a generic object is $A = (U, X, R)$, where U and X are sets and $R \subseteq U \times X$ is an usual set-theoretic relation. A morphism from A to $B = (V, Y, S)$ is a pair of functions $f : U \to V$ and $F : Y \to X$ such that $uRF(y)$ implies $f(u)Sy$. This category has products and coproducts, initial and terminal objects; the reader may find the details in [24].

The category $\mathrm{Dial}_2(\mathbf{Sets})$ has, also, a symmetric monoidal closed structure, which makes it a model of (exponential-free) intuitionistic multiplicative linear logic. Exponentials or modalities can also be modelled, as explained in the references, but they will not play a major role in what follows.

2.3 The category $\mathcal{PV}$

Vojtáš ([30]), in order to investigate certain relations between explicit objects of Analysis, introduced a category that he called *GT* (for Galois–Tukey connections). This category is a variant of the Dialectica category $\mathrm{Dial}_2(\mathbf{Sets})$ (in fact, it is a variant of the opposite (dual) category $\mathrm{Dial}_2(\mathbf{Sets})^{op}$), this was the insight of Blass in [3], and we call this category $\mathcal{PV}$ as does Blass ($\mathcal{P}$ after de Paiva, $\mathcal{V}$ after Vojtáš).

Definition 2.1
The category $\mathcal{PV}$ has, as objects, objects of $\mathrm{Dial}_2(\mathbf{Sets})$ – except that

1. All the sets U, X considered have cardinality at most the cardinality of the real numbers $\mathbb{R}$.

2. The MHD (for Moore, Hrušák and Džamonja, [19]) conditions hold: for all u in U there is an x in X such that uRx and for all x in X there is u in U such that $\neg uRx$.

The category $\mathcal{PV}$ has as morphisms certain pairs of functions, in opposite directions, as described below. ∎

If, as Blass [3], one interprets the elements of U as "questions", the elements of X as "answers" and uRx as the assertion "x answers question u", then the MHD conditions (which are kind of *non-triviality conditions*) say that "every question has an answer – but there is no particular answer that responds simultaneously all the questions".

As every intuitive interpretation of a formal definition, the preceding "translation" of the MHD conditions to a sentence on questions and answers should be taken with a pinch of salt. It is clear that the first MHD condition, which is

$$(\forall u \in U)(\exists x \in X)[uRx]$$

indeed asserts, in the expected interpretation, that "every question has an answer". But consider the second MHD condition, which is:

$$(\forall x \in X)(\exists u \in U)[\neg uRx]$$

To say that the preceding formula asserts that "there is not a particular answer that responds simultaneously to all the questions" , one should assume that such formula is equivalent to

$$\neg(\exists x \in X)(\forall u \in U)[uRx]$$

and, even considering that this is clearly true within Classical Logic, such assumption could bring some problems if we are interested in working in some more constructive environment (such as Intuitionistic Logic for example) – where the equivalence we have just referred to is no longer valid.

Even the formula $(\forall x \in X)(\exists u \in U)[\neg uRx]$ (i.e., the second MHD condition) is somehow problematic if we look at it with the eyes of a computer scientist. It asserts that, given a particular "answer" x, then *there is* a question which is not responded by x – however, in principle there are no canonical choices or constructive algorithms here which produce such "question" u which is not answered by x. The authors believe that the issue of the non-constructivity of the MHD conditions deserves further investigation. Despite their non-constructive aspects, one should notice that these so-called "non-triviality MHD conditions" are in the one hand perfectly sensible for the application that we consider (investigating cardinal invariants), but on the other hand they are not very categorical in nature – and this deserves some further investigation.

Recall that $\mathcal{PV}$ is a version of the *opposite* of $\text{Dial}_2(\mathbf{Sets})$, and therefore morphisms in $\mathcal{PV}$ satisfy the dual condition – that is, a pair of functions (f, F) is a morphism from (V, Y, S) to (U, X, R) if $f : U \to V$, $F : Y \to X$ and $f(u)Sy$ implies $uRF(y)$. The set-theoretical applications of $\mathcal{PV}$ are mostly based on the Galois-Tukey pre-order induced by the morphisms. It is somewhat perverse that here, in contrast to usual categorical logic, if o_1 and o_2 are objects of $\mathcal{PV}$ then

$$o_1 \leqslant_{GT} o_2 \iff \text{There is a morphism from } o_2 \text{ to } o_1.$$

If we interpret, given an object (U, X, R) of $\mathcal{PV}$, elements of U and X as, respectively, "problems" and "solutions" (instead of "questions" and "answers" , as done previously) and if, accordingly, we interpret uRx as the assertion "x solves u" , then the described pre-order (that is, the existence of a morphism) captures the idea of *reduction* of problems ([6]): if $o_1 \leqslant_{GT} o_2$, then problems in o_1 can be reduced to problems in o_2 – and in this sense the complexity of the problems in o_1 is smaller than or equal to the complexity of the problems in o_2, or, in other words, o_1 is "simpler than" o_2 ([30]). Indeed: consider $o_1 = (U_1, X_1, R_1)$ and $o_2 = (U_2, X_2, R_2)$ satisfying $o_1 \leqslant_{GT} o_2$, and let (f, F) be a morphism from o_2 to o_1. Let $u_1 \in U_1$ be an instance of a problem in U_1. We can reduce the solution of the problem u_1 in U_1 to the solution of the problem $f(u_1)$ in U_2, since, if $x_2 \in X_2$ solves $f(u_1) \in U_2$ then $F(x_2) \in X_1$ solves $u_1 \in U_1$ – since $f(u_1)R_2x_2 \Longrightarrow u_1R_1F(x_2)$, by the definition of the morphisms.

In the next subsection, we present an application of the category $\mathcal{PV}$ in Set Theory which clearly illustrates this idea of reduction between objects.

2.4 An Application in Set Theory: Parametrized Diamond Principles

Here we will briefly describe the first application of objects and morphisms of $\mathcal{PV}$ in Set Theory that we are aware of, as reported in [20]. Such application is in the context of combinatorial principles which are understood as weak versions of *Jensen's Diamond* – a very sophisticated consistent combinatorial principle, usually denoted by $\diamondsuit$. For the sake of completeness, we give some information on the $\diamondsuit$ principle (but none of this is essential for understanding what comes next). This combinatorial principle is a "guessing principle" which is stronger than **CH**, holds under the Axiom of Constructibility and it is often viewed as the crystallization of Jensen's combinatorial argument used (in the early 70's) in his celebrated proof that there is a Souslin tree in the constructible universe (see [16]).

For every object $o = (U, X, R)$ in $\mathcal{PV}$, Moore, Hrušák and Džamonja have introduced, more recently ([19]), the *weak parametrized diamond principle* $\Phi(o)$, which corresponds to the following statement:

Definition 2.2 (The weak parametrized diamond principle $\Phi(o)$)
For every function F with values in U, defined in the binary tree of height ω_1, there is a function g which is an "oracle", that is, there is a function $g : \omega_1 \to X$ such that g "guesses" every branch of the tree, meaning that for all $f \in {}^{\omega_1}2$ the set given by $\{\alpha < \omega_1 : F(f \restriction \alpha)Rg(\alpha)\}$ is stationary. ∎

In the preceding definition, the notion of *stationary set* refers to a subset of ω_1 that intersects all closed, unbounded subsets of ω_1 – these sets carry analogous properties to sets of positive measure, since it is a well-known fact among set theorists (see, e.g., [17], Chapter II) that the countable union of non-stationary sets is non-stationary. In general, guessing principles assert the existence of a certain "oracle" over ordinals in ω_1, which will always guess correctly at least stationarily many times – meaning that the set of ordinals on which the "oracle" intersects some set of "desired ocurrences" turns out to be stationary.

If $o = (U, U, R)$, then $\Phi(U, U, R)$ is usually written as $\Phi(U, R)$.It is worth remarking that, even before the introduction of the preceding definitions and notations, Devlin and Shelah had already shown in the late 70's ([8]) that $\Phi(2, =)$ is equivalent to the inequality "$2^{\aleph_0} < 2^{\aleph_1}$".

As can be easily verified (cf. [27]),

$$\text{If } o_1 \leqslant_{GT} o_2, \text{ then } \Phi(o_2) \Rightarrow \Phi(o_1).$$

The preceding fact is an example of what we have called *reduction* in this context: the existence of a morphism from o_2 to o_1 implies that problems concerning o_1 (in this case, the existence of an "oracle" function g as above) could be reduced to problems concerning o_2, and under this point of view the preceding implication is quite natural. We invite the readers to just apply the definitions in the most natural way and check the preceding fact.

It follows that morphisms between objects can be used to prove implications between weak diamonds. In fact, almost all of the following implications (which hold for every object o) are verified via morphisms:

$$\diamondsuit \Leftrightarrow \Phi(\mathbb{R},=) \Rightarrow \Phi(o) \Rightarrow \Phi(\mathbb{R},\neq) \Leftrightarrow \Phi(2,\neq) \Leftrightarrow \Phi(2,=) \Leftrightarrow 2^{\aleph_0} < 2^{\aleph_1}.$$

The implications "$\Phi(\mathbb{R},=) \Rightarrow \Phi(o) \Rightarrow \Phi(\mathbb{R},\neq)$ for every o" are consequences of the minimality of $(\mathbb{R},\mathbb{R},\neq)$ and of the maximality of $(\mathbb{R},\mathbb{R},=)$ in the order $\leqslant_{GT}$ over the objects of $\mathcal{PV}$; indeed, it is easy to argue that instances of problems from $(\mathbb{R},\mathbb{R},\neq)$ are the less complicated ones to solve (with plenty of solutions) and that instances of problems from $(\mathbb{R},\mathbb{R},=)$ are the more complicated ones to solve (with only one solution).

Proofs, details and/or references for all preceding implications may be found in either [19] or [27]. Notice that this sequence of implications and equivalences justify why these combinatorial principles are known as *weak diamonds* – being the inequality $2^{\aleph_0} < 2^{\aleph_1}$ the weakest diamond of all.

In a number of papers, the first author (either alone or in collaboration with Charles Morgan) has applied successfully the theory of parametrized diamond principles in order to obtain consistency and independence results (relatively to **ZFC**) in Set Theoretical Topology (see [20], [21], [28] and [22]).

3 Cardinal invariants of the continuum

3.1 Between $\aleph_0$ and $\mathfrak{c}$

One of the most investigated topics nowadays in Set Theory is the study of *cardinal invariants of the continuum*, also called *cardinal characteristics of the continuum* – or even *small cardinals*. Vaughan defined small cardinals in the following schematic way: "A small cardinal is a cardinal number that is defined as the cardinality of a set that is associated in some way with the set of natural numbers" ([29]). Indeed, the most common way to define one of such cardinals is considering the minimal cardinality of a subfamily of either $\mathcal{P}(\omega)$ or ${}^{\omega}\omega$ which does not satisfy some property which can only fail for uncountable subfamilies. Recall that, for a set theorist, $\mathcal{P}(\omega)$, ${}^{\omega}\omega$ and $\mathbb{R}$ are pretty much the same object, and, for a topologist, ${}^{\omega}\omega$ (endowed with the Tychonoff topology) is the Baire Space – a topological space which is homeomorphic to the set of irrational numbers, as a subspace of $\mathbb{R}$, and also a standard example of a Polish space (i.e., a separable and completely metrizable topological space). So, even being defined combinatorially – which facilitates the task of establishing consistency results about them –, it should not be surprising at all (but, nevertheless, it still is!) that such cardinals have a considerable influence on a number of issues from Topology and Analysis.

Blass has used the terminology *nearly countable cardinals* ([4]) to name these cardinals. Such cardinals are amazing tools when it comes to determining exactly when cardinals between $\aleph_0$ and $\mathfrak{c}$ change their behaviour.

...One of set theory's first contributions to the rest of mathematics, and still one of the

> most important, is the distinction between different infinite cardinalities, especially between countable infinity and the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum. This distinction made possible the theory of Lebesgue measure (where countable additivity is an essential ingredient but continuum additivity is impossible) and the Baire category theorem (where again "countable" clearly cannot be replaced with $\mathfrak{c}$) ...
>
> ... In these and similar situations, it is reasonable to ask where the transition from countable-like to continuum-like behaviour occurs. Of course, if one believes the continuum hypothesis, under which countable infinity $\aleph_0$ and the continuum $\mathfrak{c}$ are consecutive cardinals, then there is nothing more to be said. But if, as is known to be consistent with the usual axioms of set theory (ZFC), there are cardinals strictly between $\aleph_0$ and $\mathfrak{c}$, then it makes sense to ask whether these cardinals behave like $\aleph_0$ or like $\mathfrak{c}$ with respect to additivity of Lebesgue measure, the Baire category theorem, ..., etc. Questions of this sort are studied in the theory of *cardinal characteristics of the continuum.*

(A. Blass, 1996 [4])

The standard references for cardinal invariants of the continuum are [7] (for a more general point of view, which includes Analysis) and [9] (for applications in General Topology).

These cardinals are usually defined in terms of the combinatorial structure of functions from ω to ω, or that of infinite subsets of ω. Let us present two examples of small cardinals; the reader will realize, before the end of the paper, that our choice of examples was not a random one.

DEFINITION 3.1 (Combinatorics of functions)

1. If $f, g \in {}^{\omega}\omega$, we say that g *eventually dominates* f (and we write $f \leqslant^* g$) if $\{n < \omega : g(n) < f(n)\}$ is a finite set.
2. $\mathfrak{d}$ is the minimum number of functions $\omega \to \omega$ needed to eventually dominate every such function;
3. $\mathfrak{b}$ is the minimum number of functions $\omega \to \omega$ such that no single function eventually dominates them all. ∎

It is easy to check that $\leqslant^*$ is a pre-order.

A family $\mathcal{D}$ of functions as in 2 (that is, a family $\mathcal{D} \subseteq {}^{\omega}\omega$ such that for every $f : \omega \to \omega$ there is $g \in \mathcal{D}$ such that $f \leqslant^* g$) is said to be a *dominating family*, and a family $\mathcal{B}$ of functions as in 3 (that is, a family $\mathcal{B} \subseteq {}^{\omega}\omega$ such that for every $f : \omega \to \omega$ there is $g \in \mathcal{B}$ such that $g \not\leqslant^* f$) is said to be an *unbounded family.* So, $\mathfrak{b}$ is the smallest possible size of an unbounded family and $\mathfrak{d}$ is the smallest possible size of a dominating family.

A standard diagonal argument shows that $\mathfrak{b}$ is uncountable, and obviously one has $\mathfrak{b} \leqslant \mathfrak{d}$ (since a dominating family is clearly unbounded).

Summing up, we have the following provable sequence of inequalities involving the small cardinals $\mathfrak{b}$ and $\mathfrak{d}$:

$$|\omega| = \aleph_0 < \aleph_1 \leqslant \mathfrak{b} \leqslant \mathfrak{d} \leqslant 2^{|\omega|} = 2^{\aleph_0} = \mathfrak{c}$$

Such cardinals are indeed ubiquitous in Topology and Analysis. For instance, $\mathfrak{d}$ is the smallest number of compact subsets of the irrationals needed to cover the irrationals, identified with the Baire space ([9]). Also, if $\mathfrak{b} = \aleph_1$ then there is a

Lindelöf space X such that the product $X \times {}^{\omega}\omega$ is not normal (this result is due to E. Michael; see details and references in [29] or [9]).

In the next subsection, we will see that a number of cardinal invariants of the continuum may be associated to objects in $\mathcal{PV}$ – and, moreover, that one can use morphisms of $\mathcal{PV}$ to prove inequalities between such cardinal invariants; this constitutes the application of $\mathcal{PV}$ to Set Theory we investigate in this paper.

3.2 Cardinal invariants and the norm of objects in $\mathcal{PV}$

From this point on, we move definitely to the cardinals, and we unify the discussion on cardinal invariants and on objects and morphisms of the category $\mathcal{PV}$. Indeed, to every object in $\mathcal{PV}$ one can assign a certain cardinal invariant.

Definition 3.2 (The norm of an object of $\mathcal{PV}$)
Given an object $o = (U, X, R)$ in $\mathcal{PV}$, its *norm* (or *evaluation*) is a cardinal number, denoted $||(U, X, R)||$, defined as follows:

$||(U, X, R)||$ is the minimum cardinality of a subset $Y \subseteq X$ such that
for every $u \in U$ there is $y \in Y$ such that uRy. ■

Informally, we will refer to such cardinals as *"Abelard and Eloise cardinals"* – given the remarkable presence of the quantifiers $\forall\exists$ in their definitions.

Notice that the first MHD condition ensures that X itself satisfies the requirement for the above Y, so such cardinal is well-defined; and the second MHD condition ensures that the norm is not 1 – it should be clear that, *within Set Theory*, there is no interest in the cases where the norm is a finite cardinal, that is, a natural number. We will presently interpret the second MHD condition in a restricted case on which its role will be more intuitive.

Of course, whenever they are well-defined, we could consider norms of objects in $\mathrm{Dial}_2(\mathbf{Sets})^{op}$ as well, not only in $\mathcal{PV}$.

Inequalities between cardinals like these may be proved just by exhibiting morphisms - in an elegant and mnemonic way.

Theorem 3.3 ("Folklore" ; cited in [5])
If $o_1 \leqslant_{GT} o_2$ then $||o_1|| \leqslant ||o_2||$.

Proof. Let (φ, ψ) be a morphism from the object $o_2 = (A_2, B_2, E_2)$ to the object $o_1 = (A_1, B_1, E_1)$, and let $Y_2 \subseteq B_2$ be a minimal witness, i.e., $||o_2|| = |Y_2|$ and Y_2 is as expected.

Then, you just have to pick $Y_1 = \psi[Y_2] \subseteq B_1$ and you will have

$$(\forall x \in A_1)(\exists y \in Y_1)[xE_1y]$$

(Let $y = \psi(b)$ for some $b \in Y_2$ such that $\varphi(x)E_2b$; recall that Y_2 is as expected.)
So, Y_1 is a witness, and therefore $||o_1|| \leqslant |Y_1| \leqslant |Y_2| = ||o_2||$. ■

In the literature, as far as we know, the preceding property was exhaustively applied - but never explained in its full generality. The method of morphisms has been described in detail over the years, however no one seems interested in the reason why the method works as well as it does. Indeed, Blass said that it was *an empirical fact* that the proofs of inequalities between cardinal invariants of the continuum, *in general*, could be reduced to the representation of such cardinal invariants as norms of suitable objects of $\mathcal{PV}$ and immediately proceeded with the exhibition of morphisms. We will come back to this empirical fact presently. Before that, we will see in the next subsection that, even in a much more general theory which does not appear to be directly related to our discussion – namely, the theory of pre-orders –, some natural cardinal invariants are "Abelard and Eloise cardinals" and so they can be expressed as norms of objects of $\mathrm{Dial}_2(\mathbf{Sets})^{op}$ (or even of $\mathcal{PV}$, depending on the restriction on the cardinalities and on the validity of the MHD conditions).

3.3 A Study Case – Unboundedness and Domination in Pre-orders

The notions of unboundedness and domination, that we have just defined for the family of functions from ω into ω, can be defined for *pre-orders* in general (i.e., *not necessarily related to the set of natural numbers*). Similarly, we can define general pre-order versions of $\mathfrak{b}$, $\mathfrak{d}$.

DEFINITION 3.4 (The cardinals $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$)
Let $(\mathbb{P}, \leqslant)$ be a pre-order without a maximum element.

1. $B \subseteq \mathbb{P}$ is *unbounded* if
$$(\forall x \in \mathbb{P})(\exists y \in B)(y \nleqslant x)$$
2. $D \subseteq \mathbb{P}$ is *dominating* if it is cofinal, i.e.,
$$(\forall x \in \mathbb{P})(\exists y \in D)(x \leqslant y)$$
3. $\mathfrak{b}(\mathbb{P}) = \min\{|B| : B \subseteq \mathbb{P} \text{ is unbounded}\}$.
4. $\mathfrak{d}(\mathbb{P}) = \min\{|D| : D \subseteq \mathbb{P} \text{ is dominating}\}$. ∎

So, the previously defined small cardinals $\mathfrak{b}$ and $\mathfrak{d}$ are encompassed by this definition, since $\mathfrak{b} = \mathfrak{b}({}^{\omega}\omega, \leqslant^*)$ and $\mathfrak{d} = \mathfrak{d}({}^{\omega}\omega, \leqslant^*)$.

Notice also that, if $(\mathbb{P}, \leqslant)$ is a pre-order with a maximum element, then every subset of $\mathbb{P}$ is bounded (and so, $\mathfrak{b}(\mathbb{P})$ cannot be defined) and $\mathfrak{d}(\mathbb{P}) = 1$ (since the singleton of the maximum element will be a dominating subset of $\mathbb{P}$). It follows that requiring that the pre-order has no maximum element avoids a trivial, non-interesting case and also allows us to, at once:

1. Have both cardinals $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$ well-defined; and, moreover,
2. Look at the triple
$$(\mathbb{P}, \mathbb{P}, \leqslant)$$
as an object of $\mathrm{Dial}_2(\mathbf{Sets})^{op}$ and check that the second MHD condition holds – which, in this particular case, corresponds to the fact that $\mathbb{P}$ is unbounded in itself.

The first MHD condition, in this case, is trivial since $\leqslant$ is a reflexive relation, but it is interesting to point out that *the reflexivity of* $\mathbb{P}$ *together with the assumption of non-existence of a maximum element correspond to the MHD non-triviality conditions*, and so if the pre-order $\mathbb{P}$ has size $\leqslant \mathfrak{c}$ we will be able to look at the triple $(\mathbb{P}, \mathbb{P}, \leqslant)$ as an object of $\mathcal{PV}$.

The attentive reader has already noticed that this is the particular case we have promised earlier, where the MHD conditions have a natural and intuitive interpretation. It would be probably nice enough to have the theory on pre-orders giving us a nice interpretation of MHD conditions; but in fact, we have more than that. It is a simple formal exercise to check that $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$ are both Abelard and Eloise cardinals; more precisely,

$$\mathfrak{d}(\mathbb{P}) = ||(\mathbb{P}, \mathbb{P}, \leqslant)||; \text{ and}$$

$$\mathfrak{b}(\mathbb{P}) = ||(\mathbb{P}, \mathbb{P}, \not\geqslant)||.$$

This is a first example of the phenomenon that *it is quite usual that many natural cardinal invariants of Set Theory may be expressed as norms of objects in versions of Dialectica.*

4 Blass' empirical fact

Now we are able to present our *Main Question* – meaning that the phenomenon we would like to understand in its full depth is the one described in what follows. We have already commented on Blass' empirical fact; let us quote him properly.

> It is an empirical fact that proofs of inequalities between cardinal characteristics of the continuum usually proceed by representing the characteristics as norms of objects in $\mathcal{PV}$ and then exhibiting explicit morphisms between those objects.

(A. Blass, 1995 [3])

The main purpose of the da Silva/de Paiva collaboration is to answer the following:

Question 4.1 (The Main Question)
Why does that happen? Why cardinal characteristics of the continuum seem to behave as they were part of the $\mathcal{PV}$ category?

So, one of the goals of this research is to circumscribe the effectivity of this method and to discover why it works as well as it does; as scientists, we believe that empirical facts must have a reasonable explanation.

Indeed, in all known references (at least those known to the authors so far), all the described phenomena (representing these cardinals as norms and showing the inequalities via morphisms) are presented, described and treated almost as if they were *"facts from Nature"* – more or less in the sense of what Blass said about "the empirical fact". There are no efforts in the direction of explaining why does this happen. We would like to give such an explanation.

In what follows, we will present some *conjectural principles*, some of them with *confirming evidence*, for what we believe to be the first stones of the path towards the answer to our "Main Question".

4.1 Conjectural Principles

Let us present our first two preliminary suggestions of principles.

Conjectural Principle 4.2 (The First Conjectural Principle)
Most of the known cardinal invariants of Set Theory are "∀belard and ∃loise cardinals" – i.e., they may be represented as norms of objects in $\mathrm{Dial}_2(\mathbf{Sets})^{op}$, or even in $\mathcal{PV}$.

So, not only the morphisms are important when it comes to prove inequalities, but *equally* the possibility of representation of such cardinals as norms.

Conjectural Principle 4.3 (The Second Conjectural Principle)
The cardinals $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$, where $\mathbb{P}$ is a pre-order, are good archetypes for understanding what a cardinal invariant from Set Theory tries to capture.

That is, we believe that pre-orders may be regarded as *archetypal structures* – a kind of "safe laboratorial environment" for our investigations.

In the next subsection, we present another study case as an evidence in favour of these first two conjectural principles.

4.2 Another study case – Combinatorics of Ideals

As a confirming evidence for the first two conjectural principles, let us consider the following study case: the so-called *combinatorics of ideals* ([15]).

There is much research done on cardinals defined in terms of *ideals*.

Definition 4.4 (Ideals over a set)
Let X be a non-empty set. A family $\mathcal{I}$ of subsets of X is said to be a *(proper) ideal* if the following clauses hold:

1. $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$;
2. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$;
3. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

The dual notion is the notion of *(proper) filter* – meaning that if $\mathcal{I}$ is an ideal then the complements of the elements of $\mathcal{I}$ constitute a filter, and conversely. Intuitively, ideals are formed by *small* sets – while filters are formed by *large* sets. Indeed, if X is an infinite set, the easiest examples for these notions are, of course, the ideal of finite sets and the filter of cofinite sets. As singletons are always expected to be small, it follows that the only cases of interest for Set Theory are those where X is infinite.

When it comes to cardinals defined in terms of ideals, the ones related to the ideals $\mathcal{M}$ and $\mathcal{L}$ – respectively, the ideal of *meager* subsets of $\mathbb{R}$ and the ideal of *null* subsets of $\mathbb{R}$ – have shown themselves quite important for both Analysis and Topology; indeed, some topics cannot be even discussed without considering them. Notice that both ideals are *σ-complete* – that is, they are closed under countable unions.

Let us define the usual cardinal invariants related to ideals. For any given ideal $\mathcal{I}$ of subsets of an infinite set X, one can define the following cardinal invariants:

Definition 4.5 (Cardinal invariants related to ideals)
Let $\mathcal{I}$ be an ideal of subsets of an infinite set X. We say that

1. $\mathrm{add}(\mathcal{I})$ (the *additivity* of $\mathcal{I}$) is the smallest size of a subfamily of $\mathcal{I}$ whose union is not in $\mathcal{I}$.
2. $\mathrm{non}(\mathcal{I})$ (the *uniformity* of $\mathcal{I}$) is the smallest size of a subset of X which is not in $\mathcal{I}$.
3. $\mathrm{cov}(\mathcal{I})$ (the *covering number* of $\mathcal{I}$) is the smallest size of a subfamily of $\mathcal{I}$ whose union is X.
4. $\mathrm{cof}(\mathcal{I})$ (the *cofinality* of $\mathcal{I}$) is the smallest size of a subfamily of $\mathcal{I}$ which is cofinal in $\mathcal{I}$. ∎

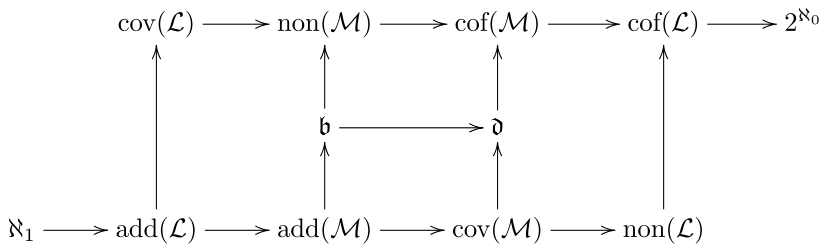
Note that $\mathrm{add}(\mathcal{I})$ is a regular cardinal, and, if $\mathcal{I}$ is σ-complete, it is clearly an uncountable cardinal. And if, besides σ-completeness, one formally assumes that $\mathcal{I}$ includes all singletons, then the following inequalities hold:

$$\aleph_1 \leqslant \mathrm{add}(\mathcal{I}) \leqslant \min\{\mathrm{cov}(\mathcal{I}), \mathrm{non}(\mathcal{I})\} \\ \leqslant \max\{\mathrm{cov}(\mathcal{I}), \mathrm{non}(\mathcal{I})\} \leqslant \mathrm{cof}(\mathcal{I}) \leqslant |\mathcal{I}|.$$

With a few basic manipulations, one can also conclude (and this is an evidence of the first principle) that *all of these four invariants* are norms of objects. Indeed, one can easily check (or see details in [11]) that the following equalities hold:

Fact 4.6
Let X and $\mathcal{I}$ be as in the previous definitions. Then, the following equalities hold:
$\mathrm{add}(\mathcal{I}) = ||(\mathcal{I}, \mathcal{I}, \not\supseteq)||$
$\mathrm{non}(\mathcal{I}) = ||(\mathcal{I}, X, \not\ni)||$
$\mathrm{cov}(\mathcal{I}) = ||(X, \mathcal{I}, \in)||$
$\mathrm{cof}(\mathcal{I}) = ||(\mathcal{I}, \mathcal{I}, \subseteq)||$ ∎

If we consider the twelve cardinals given by: $\aleph_1$ and $2^{\aleph_0}$; $\mathfrak{b}$ and $\mathfrak{d}$; the four cardinal invariants defined as above, in terms of $\mathcal{M}$; and the four cardinal invariants defined as above, in terms of $\mathcal{L}$, then we get to the well-known *Cichoń Diagram* ([10]), which we present below:



In the preceding diagram, the arrows represent inequalities between those cardinals which can be proved in **ZFC**; one also has that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$. These cardinals have huge influence on several fields of Mathematics, and we refer to the encyclopaedic book of Bartoszinsky and Judah ([2]) for a comprehensive investigation of the cardinals of such diagram and their relationship with a number of issues in Analysis and Topology.

In the next subsection, as an evidence of the second conjectural principle, we will see that, for very suitable (and reasonable) pre-orders, one can define very suitable (and reasonable) ideals such that all four cardinals may be expressed either as $\mathfrak{b}(\mathbb{P})$ or as $\mathfrak{d}(\mathbb{P})$.

4.3 Identifying "small" with "bounded"

Ideals capture the idea of "smallness" , meaning that elements of an ideal (for instance: meager sets, null sets) are "small" in a certain sense. In fact, we could think of ideals as a *formalization* of the notion of smallness, meaning that a certain property of subsets of a set correspond to a notion of smallness if those subsets satisfying such a notion form an ideal.

Let us define a reasonable notion of smallness over certain pre-orders. Consider a pre-order $\mathbb{P}$ which is *upward directed* – i.e., any finite subset of the order has an upper bound in the order (this is the only extra assumption we will make). For every element x of such a pre-order, let $B_x = \{y \in \mathbb{P} : y \leqslant x\}$. A subset Y of $\mathbb{P}$ is said to be *bounded* if (as usual) there is some $x \in \mathbb{P}$ such that $Y \subseteq B_x$ – i.e., Y is bounded if there is an x which is an upper bound of Y.

Assuming upward directedness, one has that $\{B_x : x \in \mathbb{P}\}$ generates an ideal of subsets of $\mathbb{P}$, meaning that the family of all subsets of $\mathbb{P}$ which are included in some B_x constitutes an ideal[1]. Let $\mathcal{I}_{\mathbb{P}}$ be such ideal: it is clear that the elements of $\mathcal{I}_{\mathbb{P}}$ are, precisely, the bounded subsets of $\mathbb{P}$, and so $\mathcal{I}_{\mathbb{P}}$ is the ideal of bounded subsets of $\mathbb{P}$.

For this ideal of bounded subsets of an upward directed order, one could ask about the values of the four cardinal invariants defined in the previous subsection. The first author, together with H. Garcia, has done all the calculations in [11]. We suggest the reader to also proceed with those calculations (but do not forget that we assume the Axiom of Choice!) in order to get to the following:

THEOREM 4.7 ([11])
Let $(\mathbb{P}, \leqslant)$ be an infinite, upward directed pre-order without a maximum element, and consider its ideal $\mathcal{I}_{\mathbb{P}}$ of bounded subsets. Then, the following equalities hold:

$$\mathrm{add}(\mathcal{I}_{\mathbb{P}}) = \mathrm{non}(\mathcal{I}_{\mathbb{P}}) = \mathfrak{b}(\mathbb{P}) \text{ and } \mathrm{cov}(\mathcal{I}_{\mathbb{P}}) = \mathrm{cof}(\mathcal{I}_{\mathbb{P}}) = \mathfrak{d}(\mathbb{P}).$$

∎

The Axiom of Choice is (at least, apparently) very much needed in the proofs of the preceding equalities; its role is *to fix witnesses* in some non-constructive way, as usual and expected.

To illustrate the ideas, and in order to proceed with some comparison later on, let us take from [11] a proof (using *only the definitions*) of a part of the previous theorem.

FACT 4.8
$\mathfrak{b}(\mathbb{P}) = \mathrm{add}(\mathcal{I}_{\mathbb{P}})$

[1] We have assumed upward directedness to ensure that the union of two bounded sets is also a bounded set. However it is an interesting fact that directedness also ensures that, for infinite pre-orders $\mathbb{P}$, $\mathfrak{b}(\mathbb{P})$ is an infinite cardinal - since all finite sets will be bounded. Moreover, the non-existence of maximum together with upward directedness ensures that for every $x \in \mathbb{P}$ there is a $y \in \mathbb{P}$ such that $x < y$.

Proof. Here goes a detailed proof of $\mathfrak{b}(\mathbb{P}) \leqslant \text{add}(\mathcal{I}_\mathbb{P})$: let $\{Y_\alpha : \alpha < \text{add}(\mathcal{I}_\mathbb{P})\}$ be a family (of minimal size) of elements of $\mathcal{I}_\mathbb{P}$ whose union is not in $\mathcal{I}_\mathbb{P}$ – that is, a minimal sized family of bounded subsets whose union is an unbounded subset. Pick (choice is needed !), for every $\alpha < \text{add}(\mathcal{I}_\mathbb{P})$, a $y_\alpha \in \mathbb{P}$ such that $Y_\alpha \subseteq B_{y_\alpha}$; it should be clear now that $\{y_\alpha : \alpha < \text{add}(\mathcal{I}_\mathbb{P})\}$ is necessarily an unbounded subset of $\mathbb{P}$ – otherwise the unbounded union of the Y_α's would be included in the set B_z for z an upper bound of $\{y_\alpha : \alpha < \text{add}(\mathcal{I}_\mathbb{P})\}$. It follows that

$$\mathfrak{b}(\mathbb{P}) \leqslant |\{y_\alpha : \alpha < \text{add}(\mathcal{I}_\mathbb{P})\}| \leqslant \text{add}(\mathcal{I}_\mathbb{P}).$$

The argument for the reverse inequality is similar, *mutatis mutandis.* ∎

In the following subsection, we will argue that this foreseeable procedure of *"getting a witness for this from a witness for that"*, say, is captured precisely by the morphisms – meaning that, after writing down the proofs using only the definitions, then all arguments can be successfully encoded by morphisms.

4.4 More conjectures and remarks

Here goes our third conjectural principle.

Conjectural Principle 4.9 (The Third Conjectural Principle)
If we believe that most of the cardinal invariants from Set Theory are naturally expressable as norms of objects of $\text{Dial}_2(\mathbf{Sets})^{op}$, then it should be also natural that the inequalities between such cardinal invariants will be established via morphisms of this category – and, indeed, such naturality shows itself by the fact that any reasonable, expected proof can be turned into a proof via morphisms.

That is, we believe that even if we try not to use morphisms in the proof of some inequality between Abelard and Eloise cardinals, then the predictable procedure of getting witnesses during the proof – as we have done when proving $\mathfrak{b}(\mathbb{P}) \leqslant \text{add}(\mathcal{I}_\mathbb{P})$ in the preceding subsection – would bring, encoded in disguise, the morphisms themselves. Let us use the very same example of the previous subsection in order to (or, at least, try to) convince the reader about that: a "morphisms proof" of $\mathfrak{b}(\mathbb{P}) \leqslant \text{add}(\mathcal{I}_\mathbb{P})$ would consist in the exhibition of a pair of functions witnessing $(\mathbb{P}, \mathbb{P}, \not\geqslant) \leqslant (\mathcal{I}_\mathbb{P}, \mathcal{I}_\mathbb{P}, \not\supseteq)$. Let

$$\varphi : x \in \mathbb{P} \mapsto B_x \in \mathcal{I}_\mathbb{P},$$
$$\psi : Y \in \mathcal{I}_\mathbb{P} \mapsto y \in \mathbb{P},$$

where y is *chosen* such that $Y \subseteq B_y$. Clearly, $Y \not\subseteq B_x$ implies $y \not\leqslant x$. So, (φ, ψ) is a morphism from $(\mathcal{I}_\mathbb{P}, \mathcal{I}_\mathbb{P}, \not\supseteq)$ to $(\mathbb{P}, \mathbb{P}, \not\geqslant)$, as desired. And it is also obvious that both functions above (the constituents of the morphism) are nothing more than "witnesses choosers", say; recall that the B_x are used in the definition of boundedness.

Furthermore, the categorical approach has also a great advantage. Given an object $o = (A, B, E)$, the *dual object* of o will be given by

$$o^* = (B, A, E^*)$$

(where bE^*a if, and only if, $\neg aEb$)[2].

For example, consider the four cardinal invariants defined in terms of ideals; it is immediate from Fact 4.6 that $\mathrm{add}(\mathcal{I})$ and $\mathrm{cof}(\mathcal{I})$ are norms of dual objects, as well as $\mathrm{non}(\mathcal{I})$ and $\mathrm{cov}(\mathcal{I})$. The cardinals $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$ are also norms of dual objects.

A modest contrapositive check shows that if (φ, ψ) is a morphism from o_2 to o_1 then (ψ, φ) is a morphism from o_1^* to o_2^*, and therefore we can say that *every proof counts as two* – i.e., the following proposition holds:

Proposition 4.10 (Every proof via morphisms counts as two)
If $o_1 \leqslant_{GT} o_2$, then $o_2^* \leqslant_{GT} o_1^*$ – and therefore $||o_2^*|| \leqslant ||o_1^*||$. ∎

So, if we prove via morphisms that

$$\mathfrak{b}(\mathbb{P}) = \mathrm{add}(\mathcal{I}_{\mathbb{P}})$$

then we have, by duality,

$$\mathfrak{d}(\mathbb{P}) = \mathrm{cof}(\mathcal{I}_{\mathbb{P}}).$$

Of course, the same happens to the equalities $\mathfrak{b}(\mathbb{P}) = \mathrm{non}(\mathcal{I}_{\mathbb{P}})$ and $\mathfrak{d}(\mathbb{P}) = \mathrm{cov}(\mathcal{I}_{\mathbb{P}})$; we may prove the first and get the second for free[3].

We are very close to the point we got to in this research programme, so far. We present our fourth (and last) conjectural principle; and it will become clear that we keep the faith in Category Theory to proceed with our search for the full understanding of Blass' empirical fact.

Conjectural Principle 4.11 (The Fourth Conjectural Principle)
Even believing that we are able to increase the comprehension of Blass' empirical fact through the investigation of pre-orders (and their intrinsic notions of unboundedness and domination), the more profound and general answers will come from Category Theory.

For instance, the first obvious attempt would be consider pre-orders as *posetal categories* – i.e., categories where arrows are unique whenever they exist, and in this case, as expected, $x \to y \iff x \leqslant y$. Various natural questions emerge.

Question 4.12
What would be the categorical features which constitute counterparts of the notions of unboundedness and domination in this case ? What would be the generalizations of these features to other categories ? Is there some general, categorical procedure of which *Blass' empirical fact* is just a particular case ?

Let us proceed with a little less obvious attempt.

As it should be clear by now, the authors believe that *a norm is a kind of dominating number* (as *its dual is a kind of unbounding number*), so it is natural to try to find

---

[2] We also ask the reader to check that, while the first MHD condition ensures that the norm $||o||$ is well-defined for every object $o \in \mathcal{PV}$, the second MHD condition does the same job for $||o^*||$.

[3] The reader may also check that the previously mentioned equalities $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$ are, in fact, dual equalities as well – so one has just to establish (via morphisms) one of them, and get the other for free.

conditions under which the existence of a certain function $f : \mathbb{P}_1 \to \mathbb{P}_2$ implies $\mathfrak{d}(\mathbb{P}_1) \leqslant \mathfrak{d}(\mathbb{P}_2)$ (for $\langle \mathbb{P}_1, \leqslant_1 \rangle$ and $\langle \mathbb{P}_2, \leqslant_2 \rangle$ pre-orders). It is easy to check the following theorem, whose proof is inspired in the usual procedure with morphisms of $\mathrm{Dial}_2(\mathbf{Sets})^{op}$:

Theorem 4.13
Let $\langle \mathbb{P}_1, \leqslant_1 \rangle$ and $\langle \mathbb{P}_2, \leqslant_2 \rangle$ be pre-orders and let $f : \mathbb{P}_1 \to \mathbb{P}_2$ be a function such that

(i) f is cofinal (i.e., $(\forall y \in \mathbb{P}_2)(\exists x \in \mathbb{P}_1)[y \leqslant_2 f(x)]$); and
(ii) $\forall x, y \in \mathbb{P}_1[f(x) \leqslant_2 f(y) \implies x \leqslant_1 y]$.

Under these assumptions, one has $\mathfrak{d}(\mathbb{P}_1) \leqslant \mathfrak{d}(\mathbb{P}_2)$.

Proof. Let $D \subseteq \mathbb{P}_2$ be a minimal sized cofinal set, i.e., D is dominating and $|D| = \mathfrak{d}(\mathbb{P}_2)$.

For every $x \in \mathbb{P}_1$, we can fix (via Axiom of Choice) a $d_x \in D$ such that $f(x) \leqslant_2 d_x$.

As f is cofinal, it is possible (again via Axiom of Choice) to fix, for every $y \in \mathbb{P}_2$, a $h(y) \in \mathbb{P}_1$ such that $y \leqslant_2 f(h(y))$.

It follows that, given $x \in \mathbb{P}_1$, we have $f(x) \leqslant_2 f(h(d_x))$, and therefore $x \leqslant_1 h(d_x)$. So $\{h(d) : d \in D\}$ is a cofinal subset of $\mathbb{P}_1$.

Thus, $\mathfrak{d}(\mathbb{P}_1) \leqslant \mathfrak{d}(\mathbb{P}_2)$. ∎

Again, consistently with our fourth conjectural principal, we would like to interpret what is happening in a categorical way:

Question 4.14
Consider the function f of the previous theorem as a *functor* between the posetal categories $\mathbb{P}_1$ and $\mathbb{P}_2$. What are, precisely, the features of such functor ? Were these features previously investigated in the literature, in a more general setting? And what would be the consequences of having such a functor between two categories in general?

The authors believe that answers to these questions could lead to insights for the explanation of Blass' empirical fact.

5 Final remarks and open questions

The authors have increased considerably their understanding of Blass' empirical fact since the beginning of their collaboration, and we have decided to share the initial results and conjectures with the community of Brazilian logicians in the XVII EBL (Brazilian Logic Meeting, Petrópolis, 2014) – and with the broader audience at the V UNILOG (Universal Logic Congress and School, Istambul, 2015). However, there is still a lot of work to be done, and both the presentation of this work at the referred events and the writing of this paper made us realize some of the directions that can be taken in the continuation of this research. We end up with a few more open questions.

Question 5.1
Check how much of the Axiom of Choice is needed for the calculations. Are weaker versions of choice enough for some applications ? What can be done only with the Axiom of Countable Choice, or the Axiom of Countable Choice for countable families of non-empty subsets of $\mathbb{R}$, or the Principle of Dependent Choices, etc.?

Besides this issue on the possible role of weaker forms of the Axiom of Choice, there is also a more profound question regarding choice.

Question 5.2
Could the "morphisms method" survive without the Axiom of Choice? What could be done in a choiceless context?

As described, the functions which constitute of the morphisms may be viewed, at least in some standard cases, as "witnesses choosers". Such witnesses are, in general, fixed using the Axiom of Choice. Is there some alternative path here? Or is the Axiom of Choice essential in this context? It would be very interesting, for example, to get to an equivalence of the Axiom of Choice in terms of the norms/morphisms language – or, more likely, in terms of the morphism language[4].

Still on the previous question, it is interesting to remark that the use of the Axiom of Choice for fixing witnesses can be entirely avoided in all cases where the sets under consideration are well-ordered, since a canonical choice is available (just take the minimal witness) – so, in particular, there is no need for the Axiom of Choice in the cases where the constituent sets of the objects are countable ones.

Problem 5.3
We would also like to attack the categorical problem of describing the structure of $\mathcal{PV}$ and Dialectica; for instance, the existence of NNO (natural number objects) in Dialectica was already addressed by the authors together with Charles Morgan ([26]).

The following question could be considered as a particular case of the previous problem, but we would like to give it some own weight.

Question 5.4
On the structures of $\mathcal{PV}$ and Dialectica: what about the usual constructions of products and co-products, are they closed under the MHD conditions? Or, in a more general fashion, how much of the MHD conditions could survive in a purely categorical approach?

The question about the non-constructible aspects of the second MHD condition (see discussion right after the definitions in Subsection 2.3) is included in the previous question.

Finally, we present the following question:

Question 5.5
There are notions of Category Theory which carry some similarities with Dialectica Categories. Obvious examples include the well-known Galois connections (see, e.g.,

[4] In a choiceless environment, we would not be able to consider that every set has a cardinal which is an aleph, but it would be still possible to consider statements on injective and/or surjective functions; recall also that the Schröder-Bernstein-Cantor holds in **ZF**. So, even without speaking properly about norms as cardinals, we could still speak about morphisms and equipotence (or domination) between subsets of sets in certain triples.

Examples 6.26 of [1]) or the Chu Constructions (see [25]). Do these notions have something to say about Blass' empirical fact?

Acknowledgements. The authors would like to thank the anonymous referees for a number of comments, suggestions and criticisms that have considerably improved the presentation of this work.

The first author wishes to acknowledge Joan Bagaria for a number of comments and remarks made by the occasion of a visit to the University of Barcelona in the beginning of 2015 (financially supported by MaToMUVI Project, FP7-PEOPLE-2009-IRSES Marie Curie, GA-2009-247584). Both authors would like to thank Charles Morgan for several interesting discussions.

References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker. *Abstract and concrete categories: The Joy of Cats.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1990.

[2] Tomek Bartoszyński and Haim Judah. *Set theory: On the structure of the real line.* A K Peters, Ltd., Wellesley, MA, 1995.

[3] Andreas Blass. Questions and Answers – A Category Arising in Linear Logic, Complexity Theory, and Set Theory. In *Advances in Linear Logic (ed. J.-Y. Girard, Y. Lafont, and L. Regnier), London Math. Soc. Lecture Notes 222*, pages 61–81, 1995.

[4] Andreas Blass. Nearly countable cardinals. Unpublished notes, available at http://www.math.lsa.umich.edu/ ablass/set.html, 1996.

[5] Andreas Blass. Propositional Connectives and the Set Theory of the Continuum. *CWI Quarterly (Special issue for SMC 50 jubilee), 9*, pages 25–30, 1996.

[6] Andreas Blass. Reductions between cardinal characteristics of the continuum. In T. Bartoszyński and M. Scheepers, editors, *Set theory. Annual Boise extravaganza in set theory (BEST) conference, 1992/1994, Boise State University, Boise, ID, USA*, pages 31–49. Providence, RI: American Mathematical Society, 1996.

[7] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In M. Foreman, A. Kanamori, and M. Magidor, editors, *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[8] Keith J. Devlin and Saharon Shelah. A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel Journal of Mathematics*, 29:239–247, 1978.

[9] Eric K. van Douwen. The integers and topology. In Kenneth Kunen and Jerry E. Vaughan, editors, *Handbook of set-theoretic topology*, pages 111–167. North-Holland, Amsterdam, 1984.

[10] D.H. Fremlin. Cichoń's diagram. Publ. Math. Univ. Pierre Marie Curie 66, Sémin. Initiation Anal. 23ème Année-1983/84, Exp. No.5, 13 p. (1984), 1984.

[11] Harlen Garcia and Samuel G. da Silva. Identifying *small* with *bounded*: on the ideal of bounded subsets of a directed pre-order and their cardinal invariants. Submitted, 2016.

[12] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50, no 1:1–102, 1987.

[13] Kurt Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. In *Dialectica*, pages 280–287, 1958.

[14] W.T. Gowers. The importance of mathematics. Millennium 2000 lecture, Collège de France. Available at www.dpmms.cam.ac.uk/ wtg10/importance.pdf, 2000.

[15] Michael Hrušák. Combinatorics of filters and ideals. In L. Babinkostova, A. Caicedo, S. Geschke, and M. Scheepers, editors, *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 29–69. Amer. Math. Soc., Providence, RI, 2011.

[16] R.Björn Jensen. The fine structure of the constructible hierarchy. *Annals of Mathematical Logic*, 4:229–308, 1972.

[17] Kenneth Kunen. *Set theory: An introduction to independence proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980.

[18] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1971.

[19] Justin Moore, Michael Hrušák, and Mirna Džamonja. Parametrized $\diamondsuit$ principles. *Trans. Amer. Math. Soc*, 356, no 6:2281–2306, 2004.

[20] Charles Morgan and Samuel G. da Silva. Almost disjoint families and "never" cardinal invariants. *Commentationes Mathematicae Universitatis Carolinae*, 50(3):433–444, 2009.

[21] Charles Morgan and Samuel G. da Silva. Covering properties which, under weak diamond principles, constrain the extents of separable spaces. *Acta Mathematica Hungarica*, 128(4):358–368, 2010.

[22] Charles Morgan and Samuel G. da Silva. Selectively (a)-spaces from almost disjoint families are necessarily countable under a certain parametrized weak diamond principle. *Houston Journal of Mathematics*, 2016. To appear.

[23] Valeria de Paiva. A dialectica-like model of linear logic. In D. Pitt, D. Rydeheard, P. Dybjer, A. Pitts, and A. Poigne, editors, *Category Theory and Computer Science*, pages 341–356. Springer, 1989.

[24] Valeria de Paiva. *The Dialectica Categories*. Computer Laboratory, University of Cambridge, 1990.

[25] Valeria de Paiva. Dialectica and chu constructions: Cousins ? *Theory and Applications of Categories*, 17, no 7:127–152, 2006.

[26] Valeria de Paiva, Charles Morgan, and Samuel G. da Silva. Natural number objects in dialectica categories. *Electronic Notes in Theoretical Computer Science*, 305:53–65, 2014. Proceedings of the 8th Workshop on Logical and Semantic Frameworks (LSFA).

[27] Dimi Rangel. Aplicações de princípios combinatórios em topologia geral. MsC Dissertation. UFBA Federal University of Bahia, Brazil, 2012.

[28] Samuel G. da Silva. On the extent of separable, locally compact, selectively (a)-spaces. *Colloquium Mathematicum*, 141(2):199–208, 2015.

[29] Jerry E. Vaughan. Small cardinals. *Topology Atlas Invited Contributions*, 1(3):21–22, 1996.

[30] Peter Vojtáš. Generalized Galois-Tukey-connections between explicit relations on classical objects of real analysis. In H. Judah, editor, *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 619–643. Bar-Ilan Univ., Ramat Gan, 1993.

Received (Revised Version) November 09, 2016