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# DIALECTICA PETRI NETS

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## ABSTRACT

The categorical modeling of Petri nets has received much attention recently. The Dialectica construction has also had its fair share of attention. We revisit the use of the Dialectica construction as a categorical model for Petri nets generalizing the original application to suggest that Petri nets with different kinds of transitions can be modeled in the same categorical framework. Transitions representing truth-values, probabilities, rates or multiplicities, evaluated in different algebraic structures called lineales are useful and are modeled here in the same category. We investigate (categorical instances of) this generalized model and its connections to more recent models of categorical nets.

## 1 Introduction

Petri nets exert endless fascination over category theorists. Petri nets are only one of the many modeling languages for the description of distributed systems used by computer scientists, but they enjoy the distinction of being the one category theorists most write and talk about. Maybe category theorists see Petri nets as a gauntlet thrown at them, because the definition of a morphism of Petri nets is not obvious and does lead to different categories. Maybe the bipartite graphs that usually depict Petri nets look too similar to automata ones, and these are the initial sources of good categorical examples in computing. In any case, many different categorical models of Petri nets do exist and some are fundamentally different from others. One fundamental difference is whether one concentrates in the token game and the behaviour of a given Petri net or on the graphs underlying different nets. Another difference is which possible combinators relating different Petri nets one considers. The aspect we concentrate in this work is the kinds of labels that can be used in a Petri net.

A *Petri net* is simply a directed bipartite graph that has two types of elements, *conditions* and *events* (also called places and transitions). These are usually depicted as white circles and rectangles, respectively (see Figure 1). A place can contain any number of tokens, depicted as black circles. Over this fixed structure of possible events and conditions, a causal dependency (or flow) relation between sets of events and conditions is described via pre- and post-relations, and it is this structure which determines the possible **dynamic** behaviour of the net. A transition in this causal dependency relation is enabled if all places connected to it as inputs contain at least one token. Finally given an initial position of tokens in a net (an initial marking) the “token game” (which we are not modelling) can be started and the system will evolve.

We explore the model originally introduced by Winskel [1, 2], but use it with morphisms, as in the work of Brown [3] and others, that relate Petri nets to constructors in Linear Logic [4]. Proofs for all propositions and the main theorem appear in Appendix A.

## 2 Petri nets and their transitions

Networked systems are determined by their connections [5]. Perhaps the most basic type of relationship in any network is one that only allows us to express either presence or absence, that is, where the relationship connecting nodes uses the set  $\{0,1\}$  as a ruler or label set. In real-world applications this is, though, not sufficient. In this section we explore frequent and rich applications of Petri nets, from chemical reaction networks to metabolic networks, searching for the kind of labels used on pre- and post- conditions.

*Chemical reaction networks.* Chemical combination is compositional in nature. Although data on substance reactivity are typically annotated as a list of chemical equations (see Figure 1a), chemists reason on the network structure (see Figure 1b) that emerges when the reactions are connected to make their concurrency explicit [6]. Synthesis planning is a prominent example: suppose we want to synthesize substance F, but we cannot carry out reaction  $r_2$ , because we have no substance C on our lab’s shelf. In such a case, equation  $r_1$  provides another synthetic route, since it is possible to obtain F from A and B (and E). In other words, the synthesis of F results from composing reactions  $r_1$  and  $r_2$ .

Directed hypergraphs and their enhancements, such as Petri nets, are used to model chemical reaction networks for they are models of concurrency of directed relations. These models provide a rich semantic basis on which to interpret questions that arise in chemistry, such as what substances can be synthesized from a given set of starting materials? [7] Given a target substance, which synthetic routes are known and which starting materials are needed to reach the target? [8] Do chemical reactions turn targets into key precursors? [5] How many synthetic routes pass through a given reaction? These questions can be answered by probing the topology and geometry of the wiring of the network. The first two questions are answered by defining suitable closure operators [7], and the last two questions by computing the curvature of the edges of the network [9], taking into consideration the proportions in which substances combine (stoichiometric coefficients).

At the level of abstraction described above, transitions of chemical reaction networks are discrete in nature, and pre- and post- conditions correspond to presence/absence or to stoichiometric coefficients, which can be model by the rulers  $\{0, 1\}$  and  $\mathbb{N}$ , respectively.

*Metabolic networks.* These networks are comprised of the metabolic pathways (network of chemical reactions) and the gene interactions that regulate them. A key aspect of the former is the kinetic modelling. There, Petri nets model reaction rates. For elementary reactions, which take place in a single step, the Law of Mass Action states that reaction rates are proportional to the concentration of reactants. Both quantities, rate of reactions and concentration of reactants, are usually taken as positive real numbers; therefore, in this application, Petri nets are challenged to handle continuous tokens and transitions, which requires the ruler  $\mathbb{R}^+$ . On the other hand, gene interactions are handled by implementing genetic switches that are modeled by discrete transitions. A Petri net model for a metabolic network therefore needs two different rulers on the same net.

When applied to concrete metabolisms, a Petri net model will usually need to incorporate more than two rulers at the same time. For instance, [10] shows a hybrid Petri net representation of the gene regulatory network of *C. elegans* that is labeled with discrete and continuous transitions, but also with negative integers, real numbers, strings, and products of them.

Summarising, applications may need rulers such as  $\{0, 1\}$ ,  $\{-1, 0, -1\}$  (for data uncertainty, which is common in complex network systems [11]),  $\mathbb{N}$ ,  $\mathbb{R}^+$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$ , strings, and their finite products. The ability of choosing from a vast pool of rulers to label pre- and post- conditions is one of the strengths of the categorical construction presented in this paper.

## 3 Petri nets via Dialectica Categories

Petri nets were described categorically in many works (e.g. [3], [12]) and are still been discussed [13], [14]. Models need to capture the practitioner’s imagination and make themselves useful, both for calculations and for insights. Categorical models can be useful for both insights and calculations, but we have not seen categorical models that encompass different kinds of transitions in a single net.

Petri nets were modeled using Dialectica categories [15] previously, but the original Brown and Gurr model [16] worked only for *elementary nets*, that is nets whose transitions are marked with  $\{0,1\}$  for presence or absence of a

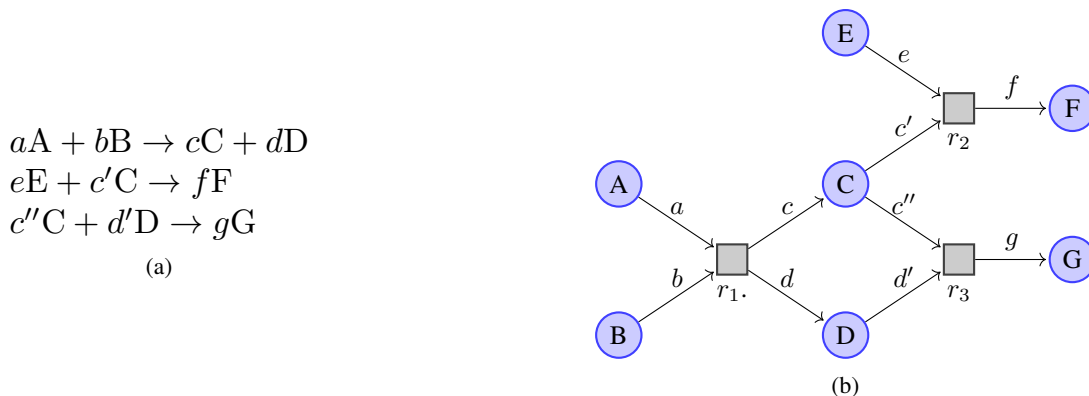


Figure 1: Petri net representation of chemical reaction data: as a list (a) and as a network (b).  $A, B, \dots, H$  are substances and  $a, b, \dots, h$  are stoichiometric coefficients that indicate the proportion in which they combine.

relationship. An extension of this modeling to deal with integers  $\mathbb{N}$  was planned [17, 18], but never published. In this work we put together different kinds of transition, in a single categorical framework. This way the categorical modelling applies to the many kinds of newer applications [19] that already use different kinds of labels on the transitions.

The original dialectica construction [15] was given in two different styles called the categories  $DC$  [20] and the categories  $GC$  [21]. For both constructions  $C$  is a cartesian closed category with some other structure. The first style is connected to Gödel’s Dialectica Interpretation hence the ‘D’ in  $DC$  for Dialectica. The second style called  $GC$  ([21]) is based on a suggestion of Girard’s (hence the ‘G’) on how to simplify the first construction, if one wants a model of Linear Logic. These two constructions are connected, via monoidal comonads as described in [15]. Here we are mostly interested in the construction called  $GC$ , whose morphisms are simpler. This construction can be explained, when the category  $C$  is  $\text{Set}$ , using two ‘apparently’ different descriptions. This is because a relation in  $\text{Set}$  between  $U$  and  $X$  can be thought of as either a subset of the product,  $\alpha \subseteq U \times X$ , or as map into 2,  $\alpha: U \times X \rightarrow 2$ .

The work here uses only the second kind of description, defining general relation maps into algebraic structures called **lineales**. This is because changing the lineale where our relations take ‘values’, gives us the possibility of modeling several different kinds of processes. The original dialectica constructions deals only with the Heyting algebra-like lineale 2. Here we discuss several other lineales and dialectica categories built over these different lineales.

### 3.1 Related work

Our work fits in the vast landscape of categorical approaches to Petri nets building on [18, 17]. Meseguer’s seminal work [12] focused on reachability properties of Petri nets, defining a category of all possible executions of a net. This work adopted the collective tokens philosophy. Its ideas were extended to the individual tokens philosophy in [22]. Other categorical models of Petri nets focus on obtaining nets by composing smaller nets along some boundaries. One of the first compositional models doing this was [23] where nets are composed along common places. In [24], nets are composed along common transitions and compositionality is used to study reachability properties of Petri nets. The work of [3], [17] and [18] concentrates on combining Petri nets via different monoidal products that give to the category of Petri nets a linear logic structure. More recently, there has been numerous works building on the ideas of [12] and adopting the formalism of [23]. In [13] and [25], the authors focus on studying the categorical properties of reachability. In [26] a more fine-grained categorical model is proposed, that allows Kock to encompass the individual and collective token philosophies in the same framework. Finally, [27] constructs a unifying framework for [12], [22] and [26] extending [25].

Our work extends the approach of [3] to allow different kinds of arcs, e.g. inhibitor, probabilistic, partially defined, natural/integer numbers valued, and the coexistence of them in the same net.

## 4 The category $M_L\text{Set}$ and its structure

In order to define our category of Petri nets, we need to explain what kind of structure, that of a *lineale*, is required on the set of truth values that we use as codomain for pre- and post-conditions relations.

## 4.1 Lineales

A lineale is a monoid together with a partial order compatible with the monoidal product, and such that every pair of elements has an internal-hom. The monoidal product, the partial order and the internal-hom are used to define the adjunction that characterizes a lineale, as this is just a poset version of a symmetric monoidal closed category.

**Definition 4.1** (Partially ordered monoid). A *partially ordered monoid*  $(L, \sqsubseteq, \otimes, e)$  is a monoid  $(L, \otimes, e)$  equipped with a partial order  $\sqsubseteq$  that is compatible with the monoidal operation, i.e. if  $a \sqsubseteq b$  and  $a' \sqsubseteq b'$  then  $a \otimes a' \sqsubseteq b \otimes b'$ .

In the setting of partially ordered monoids, internal-homs are easier to define.

**Definition 4.2** (Internal hom in a monoid). Let  $(L, \sqsubseteq, \otimes, e)$  be a partially ordered monoid. A binary operation  $\multimap: L^{op} \times L \rightarrow L$  is said to be an *internal hom* when it is right adjoint to the monoidal product  $\otimes$ , i.e.  $\forall a, b, c \in L, b \otimes c \sqsubseteq a \Leftrightarrow b \sqsubseteq c \multimap a$ . The internal hom is also required to respect the ordering, contravariantly in the first coordinate and covariantly in the second, i.e. if  $b \sqsubseteq a$  and  $a' \sqsubseteq b'$  then  $a \multimap a' \sqsubseteq b \multimap b'$ .

We can now define the central notion of this section, that of a lineale.

**Definition 4.3** (Lineale). A *lineale* is a tuple  $(L, \sqsubseteq, \otimes, e, \multimap)$  such that  $(L, \sqsubseteq, \otimes, e)$  is a partially ordered monoid and  $\multimap$  is an internal hom for  $(L, \sqsubseteq, \otimes, e)$ .

**Example 4.4.** Examples of lineales are  $(\mathbb{N}, \geq, +, 0, \multimap)$  and  $(\mathbb{R}^+, \geq, +, 0, \multimap)$ , where  $a \multimap b = \begin{cases} b - a & a \leq b \\ 0 & a > b \end{cases}$ .

Any partially ordered group (see Definition 4.5) is a lineale with  $a \multimap b = b \otimes a^{-1}$  (see Proposition 4.6). Some of our examples will fall into this case.

**Definition 4.5** (Partially ordered group). A *partially ordered group*  $(G, \sqsubseteq, \otimes, e, (-)^{-1})$  is a partially ordered monoid  $(G, \sqsubseteq, \otimes, e)$  together with an inverse operation  $(-)^{-1}$  that makes  $(G, \otimes, e, (-)^{-1})$  a group and respects the ordering contravariantly, i.e. if  $a \sqsubseteq b$  then  $b^{-1} \sqsubseteq a^{-1}$ .

**Proposition 4.6.** A *partially ordered group*  $(G, \sqsubseteq, \otimes, e, (-)^{-1})$  can be endowed with the structure of a lineale with  $a \multimap b := b \otimes a^{-1}$ .

**Example 4.7.** Examples of lineales obtained from partially ordered groups are  $(\mathbb{Z}, \geq, +, 0, -)$  and  $(\mathbb{R}, \geq, +, 0, -)$ .

## 4.2 The category $M_L\text{Set}$

Having defined a lineale, we proceed to construct the intermediate category  $M_L\text{Set}$  over which our category of Petri nets  $\text{Net}_L$  is built.

**Definition 4.8** (Category  $M_L\text{Set}$ ). Given a lineale  $(L, \sqsubseteq, \otimes, e, \multimap)$ , the category  $M_L\text{Set}$  is defined by the following data.

- An object is a triple  $(U, X, \alpha)$ , denoted by  $U \xleftarrow{\alpha} X$ , where  $U, X$  are sets and  $\alpha: U \times X \rightarrow L$  is a function in  $\text{Set}$ .
- A morphism  $(f, F): (U, X, \alpha) \rightarrow (V, Y, \beta)$  is a pair of morphisms,  $f: U \rightarrow V$  and  $F: Y \rightarrow X$  in  $\text{Set}$ , such that  $\forall u \in U \forall y \in Y \alpha(u, Fy) \sqsubseteq \beta(fu, y)$ .

$$\begin{array}{ccc}
U \times Y & \xrightarrow{f \times \mathbb{1}_Y} & V \times Y \\
\downarrow \mathbb{1}_U \times F & \sqsubseteq & \downarrow \beta \\
U \times X & \xrightarrow{\alpha} & L
\end{array}$$

The category  $M_L\text{Set}$  allows us to have  $L$ -valued relations, including multirelations ( $L = \mathbb{N}$ ) and any other label set that can be seen as a lineale.

**Proposition 4.9.**  $M_L\text{Set}$  is a category.

We can now define the structure of  $M_L\text{Set}$ . We will define maps up to symmetries in  $\text{Set}$  to avoid distracting the reader with details.

**Definition 4.10** (Product and coproduct in  $M_L\text{Set}$ ). Given two objects  $A = (U \xleftarrow{\alpha} X)$  and  $B = (V \xleftarrow{\beta} Y)$  in  $M_L\text{Set}$ , we define their cartesian product  $A \& B$  as the following object.

$$A \& B = (U \times V \xleftarrow{\alpha \& \beta} X + Y)$$

The function  $\alpha \& \beta$  is  $U \times V \times (X + Y) \xrightarrow{[\alpha \times \epsilon_V, \beta \times \epsilon_U]} L$ , where  $\epsilon_U$  is the function that discards  $U$  in  $\text{Set}$ .

Similarly, we define their coproduct  $A \oplus B$  as the following object.

$$A \oplus B = (U + V \xleftarrow{\alpha \oplus \beta} X \times Y)$$

The function  $\alpha \oplus \beta$  is  $(U + V) \times X \times Y \xrightarrow{[\alpha \times \epsilon_Y, \beta \times \epsilon_X]} L$ .

Now, we use the monoidal operation of  $L$  to define a tensor product in  $M_L\text{Set}$ .

**Definition 4.11** (Monoidal product in  $M_L\text{Set}$ ). Given two objects  $A = (U \xleftarrow{\alpha} X)$  and  $B = (V \xleftarrow{\beta} Y)$  in  $M_L\text{Set}$ , we define their monoidal product  $A \otimes B$  as the following object.

$$A \otimes B = (U \times V \xleftarrow{\alpha \otimes \beta} X^V \times Y^U)$$

Where  $X^V$  and  $Y^U$  are internal hom objects in  $\text{Set}$  and the function  $\alpha \otimes \beta$  is defined by the following composition.

$$U \times V \times X^V \times Y^U \xrightarrow{\Delta_U \times \Delta_V \times \mathbb{1}_{X^V} \times \mathbb{1}_{Y^U}} U \times U \times V \times V \times X^V \times Y^U \xrightarrow{\mathbb{1}_U \times \text{eval} \times \mathbb{1}_V \times \text{eval}} U \times X \times V \times Y \xrightarrow{\alpha \times \beta} L \times L \xrightarrow{\otimes} L$$

where  $\Delta_U$  is the diagonal map on  $U$  and  $\text{eval}$  is the evaluation map in  $\text{Set}$ . Spelling out this definition elementwise, we obtain  $(\alpha \otimes \beta)(u, v, f, g) = \alpha(u, fv) \otimes \beta(v, gu)$ .

**Proposition 4.12.** *The construction above induces a functor  $\otimes: M_L\text{Set} \times M_L\text{Set} \rightarrow M_L\text{Set}$ , which is a monoidal product on  $M_L\text{Set}$ .*

**Definition 4.13** (Internal hom in  $M_L\text{Set}$ ). Given two objects  $A = (U \xleftarrow{\alpha} X)$  and  $B = (V \xleftarrow{\beta} Y)$  in  $M_L\text{Set}$  we define their internal hom,  $[A, B]$ , as follows:

$$[A, B] = V^U \times X^Y \xleftarrow{[\alpha, \beta]} U \times Y$$

The function  $[\alpha, \beta]$  is defined by the following composition.

$$V^U \times X^Y \times U \times Y \xrightarrow{\mathbb{1}_{V^U} \times X^Y \times \Delta_U \times \Delta_Y} V^U \times X^Y \times U \times U \times Y \times Y \xrightarrow{\mathbb{1}_U \times \text{eval} \times \text{eval} \times \mathbb{1}_Y} U \times X \times V \times Y \xrightarrow{\alpha \times \beta} L \times L \xrightarrow{\multimap} L$$

Spelling out this definition elementwise, we obtain  $[\alpha, \beta](f, F, u, y) = \alpha(u, Fy) \multimap \beta(fu, y)$ .

**Proposition 4.14.** *The construction above induces a functor  $[-, -]: M_L\text{Set}^{op} \times M_L\text{Set} \rightarrow M_L\text{Set}$ .*

The category  $M_L\text{Set}$  has products and coproducts and is a symmetric monoidal closed category with the structure defined so far.

**Theorem 4.15.** *The category  $M_L\text{Set}$  has products and coproducts as in Definition 4.10 and is a symmetric monoidal closed category with monoidal product as in Definition 4.11 and internal hom as in Definition 4.13.*

## 5 A category of Petri nets

A Petri net is given by a set of places  $U$ , a set of transitions  $X$ , and has two relations between these two sets that specify the precondition relation  $\blacktriangleright \alpha$  and the postcondition relation  $\alpha \blacktriangleright$ . In our case these relations will be valued in a generic lineale  $L$  and the pre- and post- conditions will be objects in  $M_L\text{Set}$ . The category of Petri nets that we consider has Petri nets as objects and is obtained by putting together two copies of  $M_L\text{Set}$  (by taking a pullback in  $\text{Cat}$ ), one representing preconditions  $\blacktriangleright \alpha: U \times X \rightarrow L$  and the other one representing postconditions  $\alpha \blacktriangleright: U \times X \rightarrow L$ . A morphism  $(f, F): A \rightarrow B$  in this category represents the fact that the Petri net  $B$  can simulate the net  $A$  as the conditions on the morphisms in  $M_L\text{Set}$  ensure that the preconditions and the postconditions of  $A$  are ‘smaller’ than those of  $B$ :  $\forall u \in U \forall y \in Y \blacktriangleright \alpha(u, Fy) \sqsubseteq \blacktriangleright \beta(fu, y) \wedge \alpha \blacktriangleright (u, Fy) \sqsubseteq \beta \blacktriangleright (fu, y)$ .

**Definition 5.1** (Category  $\text{Net}_L$ ). Given a lineale  $(L, \sqsubseteq, \otimes, e, \multimap)$ , the category  $\text{Net}_L$  is defined by the following data.

- An object is a pair  $A = (\blacktriangleright A, A\blacktriangleright)$  of objects  $U \xrightarrow{\alpha} X$  and  $U \xleftarrow{\alpha} X$  in  $M_L\text{Set}$ .
- A morphism  $(f, F): (\blacktriangleright A, A\blacktriangleright) \rightarrow (\blacktriangleright B, B\blacktriangleright)$  is a morphism both  $(f, F): \blacktriangleright A \rightarrow \blacktriangleright B$  and  $(f, F): A\blacktriangleright \rightarrow B\blacktriangleright$  in  $M_L\text{Set}$ .

The structure of  $M_L\text{Set}$  defines analogous structure in  $\text{Net}_L$ .

**Definition 5.2** (Structure of  $\text{Net}_L$ ). The category  $\text{Net}_L$  inherits the structure of  $M_L\text{Set}$ . All the connectives are defined componentwise:

- $A \otimes B = (\blacktriangleright A \otimes \blacktriangleright B, A\blacktriangleright \otimes B\blacktriangleright)$ .
- $[A, B] = ([\blacktriangleright A, \blacktriangleright B], [A\blacktriangleright, B\blacktriangleright])$ .
- $A \& B = (\blacktriangleright A \& \blacktriangleright B, A\blacktriangleright \& B\blacktriangleright)$ .
- $A \oplus B = (\blacktriangleright A \oplus \blacktriangleright B, A\blacktriangleright \oplus B\blacktriangleright)$ .

Examples of Petri nets modelled in this category are in the next section, where we will show how, with the possibility of changing the lineale, we can encompass different kinds of nets.

## 6 Different lineales

While the lineale  $2$  is associated with Boolean and Heyting algebras (traditional algebraic models for classical and intuitionistic propositional logic) other lineales are associated with different non-classical systems. We describe a 3-valued propositional logic where the undefined truth-value, the ‘‘unknown’’ state, can be thought of as neither true nor false. The adjunction determines the structure we consider for this lineale  $3$ . We should also mention the lineale  $4$ , associated with Belnap-Dunn’s four-valued logic. (These four values also correspond to the algebraic identities for the two conjunctions and two disjunctions of Linear Logic.)

We then consider the lineale built out of natural numbers, but with the opposite order from the natural order in  $\mathbb{N}$ . Using this lineale we build the dialectica category of multirelations  $M_{\mathbb{N}}\text{Set}$  [17] based on Lawvere’s ideas about generalizing metric spaces [28].

Further we consider integers  $\mathbb{Z}$  with their usual order as a lineale. We believe this style of dialectica category can be profitably used to model systems where some transitions can cancel other transitions.

Next we consider the reals, in the form of the closed interval  $[0, 1]$ . These have long been considered for fuzzy sets, as the real number associated with a pair  $(u, x)$  can be thought of as the probability of the association between  $u$  and  $x$ . Finally, we consider the coexistence of several lineales in a single net by taking finite products of them. In possession of this collection of lineale structures, we define Petri nets using pre- and post-conditions between sets of events and conditions.

### 6.1 Original Dialectica $L = 2$

The original work on the categorical version of the dialectica interpretation has concentrated on relations that take values into  $2$ , considered as a lineale. By considering relations with values on this lineale, which correspond to ordinary relations, we obtain the elementary Petri nets, those where pre- and post-conditions only say whether or not a place is a pre- or post-condition for a transition.

### 6.2 Kleene Dialectica $L = 3$

In this section, we consider a version of the Dialectica construction where the lineale is defined on the three elements set  $3 = \{-1, 0, 1\}$ . We can think of  $-1$  and  $1$  as *false* and *true* respectively. The additional truth value  $0$  can be interpreted as *undefined*. We can equip this set with the following structure.

- The monoid structure is given by the minimum,  $a \otimes b := \min\{a, b\}$ , whose unit is  $1$ .
- The ordering is given by the usual ordering  $-1 < 0 < 1$ .
- The implication is defined to yield an adjunction with the conjunction:  
 $a \multimap b := \max\{x : x \otimes a \leq b\}$ .

Spelling out the condition for the internal hom,  $a \multimap b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$ .

We can check that this structure is indeed that of a lineale.

**Lemma 6.1.** *The three-elements set is a lineale with the structure defined above.*

Thus, we can define the dialectica construction over  $(3, \leq, \otimes, 1, \multimap)$  and Petri nets with weights in 3 accordingly.

**Example 6.2.** We take as motivating example the model of the chemical reaction regulating the circadian clock of *Synechococcus Elongatus* [29] that is composed of two successive phosphorylations and two successive dephosphorylations (which are the transitions labelled with phos. and dephos. in Figure 2). There is experimental evidence [30] for the existence of further feedback loops in this model. However, the precise underlying mechanism is still unknown. We can take into account this unknowns in our model for Petri nets by adding arcs with 0 weight (presence and absence are represented by 1 and -1 respectively). The Petri net in Figure 2 shows the values of the pre- and post- conditions relations as weights on the arcs.

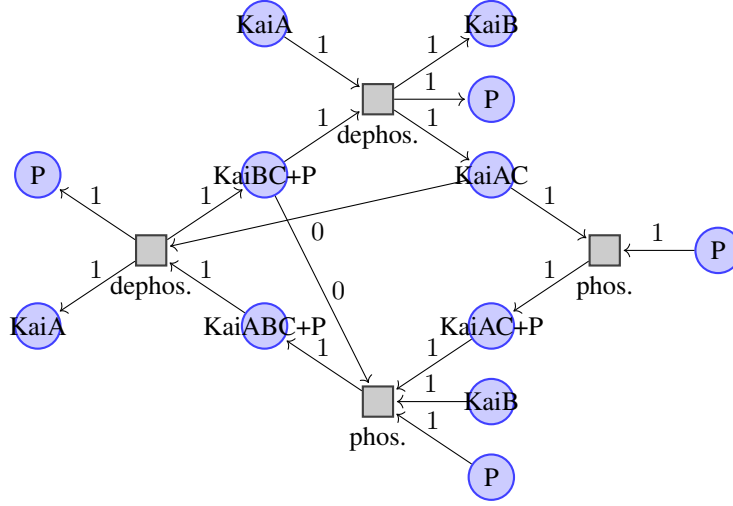


Figure 2: Petri net representing the chemical reaction network regulating the circadian clock of *Synechococcus Elongatus*. Present and undefined relations are labeled by 1 and 0, respectively.

### 6.3 Multirelation Dialectica $L = \mathbb{N}$

In this section, we consider a version of the Dialectica construction where the lineale is defined on the natural numbers  $L = \mathbb{N}$ . While we can think of the classical truth values as indicating whether or not a certain substance is present, we can think of the natural numbered truth values as indicating how much of a certain substance is present in a chemical reaction. We can equip this set with the following structure.

- The monoid structure is given by the sum of natural numbers,  $a \otimes b := a + b$ , whose unit is 0.
- The partial ordering is given by the opposite of the usual order on natural numbers.
- The implication is given by truncated subtraction:  $a \multimap b := \max\{b - a, 0\}$ .

The basic suggestion of this structure on the natural numbers comes from [28]. We can check that this structure is indeed that of a lineale.

**Lemma 6.3.** *The set of natural numbers is a lineale with the structure defined above.*

**Example 6.4.** As every chemical reaction, the one to obtain water from oxygen and hydrogen needs stoichiometric coefficients to be represented properly. We can use multirelations to take these into account, as shown in Figure 3.

### 6.4 Integers Dialectica $L = \mathbb{Z}$

Similarly to the Multirelation Dialectica, we can consider the Dialectica construction for the particular case of  $L = \mathbb{Z}$ . We endow the set of integers with a lineale structure in the following way.



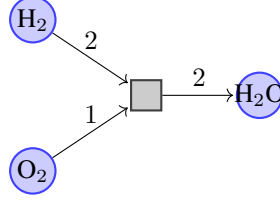


Figure 3: Petri net representing the chemical reaction  $2\text{H}_2 + \text{O}_2 \rightarrow 2\text{H}_2\text{O}$ .

- The monoid structure is given by the sum of integers,  $a \otimes b := a + b$ , whose unit is 0.
- The partial order is given by the usual ordering on the integers.
- The internal hom is given by subtraction:  $a \multimap b := b - a$ .

As  $(\mathbb{Z}, \leq, +, 0, -)$  is a partially ordered group, it is automatically a lineale with the internal hom defined above by Proposition 4.6.

**Example 6.5.** Empirical systems often need to locally reverse the logic of preconditions to express that the presence of tokens in a given place “disables” a transition. Several different concepts of inhibitor arcs can be modeled by Petri nets including the “threshold inhibitor arc”. Reaction inhibitors in chemistry illustrate the situation: in Figure 4 chemical reaction  $r$  will not take place if the amount of substance I exceeds 3, a condition that is expressed by its inverse  $-3$ .

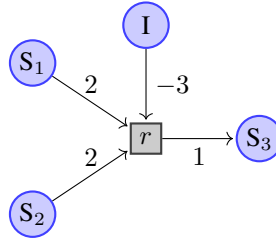


Figure 4: Petri net representation of the chemical reaction  $\text{S}_1 + \text{S}_2 \rightarrow \text{S}_3$ . The inhibitor arc is labeled by  $-3$ , expressing that 3 is the minimum amount of substance I that prevents  $r$  from taking place.

## 6.5 Probabilistic Dialectica $L = [0, 1]$

So far we considered only (topologically) discrete lineales. Now we want to discuss real numbers, in particular the closed interval  $[0, 1]$ . We first show that the closed interval  $[0, 1]$  admits a lineale structure.

- The monoid structure is given by the product of real numbers,  $a \otimes b := a \cdot b$ , whose unit is 1.
- The partial order is given by the usual ordering on the reals.
- The internal hom is given by a ‘truncated division’  $a \multimap b := \begin{cases} \frac{b}{a} & a \neq 0 \wedge a \geq b \\ 1 & a = 0 \vee a < b \end{cases}$ .

**Lemma 6.6.** *The closed interval  $[0, 1]$  is a lineale with the structure defined above.*

**Example 6.7.** The SIR (Susceptible, Infectious, Recovered) model is a simple compartmental model for infectious diseases. A susceptible individual has a contact with an infectious individual with probability  $p_c$  and, after the contact, it can be infected with probability  $p_I$ , or remain susceptible with probability  $1 - p_I$ . On the other hand, an infectious individual can recover with probability  $p_R$  or remain infectious with probability  $1 - p_R$ . This setting can be represented with a Petri net where the relations between places and transitions are valued in  $[0, 1]$  (Figure 5).

## 6.6 Product of lineales

We have produced a pool of lineales, each of them suitable for transitions taking values in certain data types ( $\{0, 1\}$ ,  $\{-1, 0, -1\}$ ,  $\mathbb{N}$ ,  $\mathbb{Z}$  or  $\mathbb{R}^+$ ). As discussed in section 2, in empirical data analysis, a transition often carries data on more



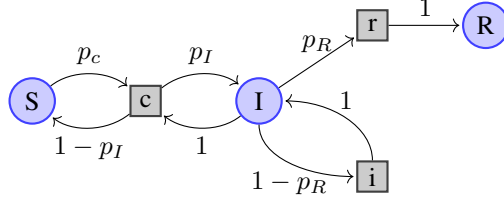


Figure 5: Petri net representing the SIR model.

than one variable simultaneously. In this section, we show that any finite combination of lineales can be endowed with the structure of a lineale by taking finite products of them.

We remark that, because lineales are just the poset-version of symmetric monoidal closed categories, the next proposition is simply an instantiation of the idea that symmetric monoidal closed categories form a cartesian category **SymClosedCat** whose objects are symmetric monoidal closed categories, and morphisms are functors that preserve the adjunction. This category is Cartesian. This seems folklore, but we could not find it in any of the usual references (e.g. [31]). We only need the poset version here and we state it as such.

**Proposition 6.8.** *If  $(L_1, \leq_1, \otimes_1, e_1, -\circ_1)$  and  $(L_2, \leq_2, \otimes_2, e_2, -\circ_2)$  are lineales, then  $(L_1 \times L_2, \leq, \otimes, e, -\circ)$  is a lineale with the following structure.*

- $l \otimes l' = (l_1 \otimes_1 l'_1, l_2 \otimes_2 l'_2)$
- $e = (e_1, e_2)$
- $l \leq l'$  if and only if  $l_1 \leq_1 l'_1$  and  $l_2 \leq_2 l'_2$
- $l -\circ l' = (l_1 -\circ_1 l'_1, l_2 -\circ_2 l'_2)$

for  $l = (l_1, l_2), l' = (l'_1, l'_2) \in L_1 \times L_2$ .

**Example 6.9.** There is a dual situation to inhibition in chemistry, namely, catalysis. A catalyst is a substance that increases the reaction rate without being consumed by the reaction. The presence of a substance S in a chemical reaction might then play one of three roles: reactant/product, inhibitor, or catalyst. We claim that the product of the lineales  $\mathbb{R}^+$  and  $\mathbb{Z}$  has enough expressive power to model reaction rates in the presence of both inhibitors and catalysts. In Figure 6 pairs of the form  $(r, 0)$ , state that those substances are not inhibitors nor catalysts, and  $r$  is the rate at which a substance is consumed or produced. The negative number in the label  $(r_4, -3)$  expresses that I is an inhibitor of reaction  $r$ , and -3 the minimum amount of I required to slow down the reaction by the rate  $r_4$ . Finally, the label  $(r_5, 5)$  indicates that C is a catalyst and 5 is the minimum amount of C required to increase the reaction rate by  $r_5$ .

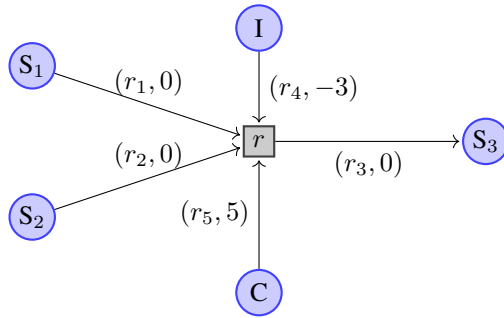


Figure 6: Petri net representation of reaction rates for the chemical reaction  $S_1 + S_2 \rightarrow S_3$  in the presence of an inhibitor I and a catalyst C. Labels are pairs  $(r, z)$  where  $z$  states the role of the substance as reactant/product (zero), inhibitor (negative integers), and catalysts (positive integers); and  $r$  the rate at which the substance is consumed/produced (if  $z = 0$ ), or at which the reaction rate increases ( $z > 0$ ) or is slowed down (if  $z < 0$ ).

## 7 Conclusions and further work

Most of the recent work on Petri nets focuses on the unfoldings of a single net. We have presented a categorical model for Petri nets that focuses on the diverse nature of network relations. This is a fundamentally different approach to Petri

nets since it allows the use of different kinds of transitions (different kinds of labels in their graphs), while maintaining their compositionality.

Our model can handle different kinds of transition whose labels can be represented as a lineale (a poset version of a symmetric monoidal closed category). Several sets of labels, from those often used in empirical data modeling, can be endowed with the structure of a lineale, including: stoichiometric coefficients in chemical reaction networks ( $L = \mathbb{N}$ ), reaction rates ( $L = \mathbb{R}^+$ ), inhibitor arcs ( $L = \mathbb{Z}$ ), gene interactions ( $L = \{0, 1\}$ ), unknown or incomplete data ( $L = \{-1, 0, 1\}$ ), and probabilities ( $L = [0, 1]$ ).

The structure of the lineale is lifted to the category  $M_L\text{Set}$  from which the category  $\text{Net}_L$  of Petri nets is built. Both  $M_L\text{Set}$  and  $\text{Net}_L$  are symmetric monoidal closed categories with finite products and coproducts, providing a compositional way to put together smaller nets into bigger ones, making sure that algebraic properties of the components are preserved in the resulting net.

The category  $\text{Net}_L$  is a model for weighted and directed bipartite relations and therefore we anticipate applications of the compositionality of  $\text{Net}_L$  in the broader context of directed bipartite graphs, in particular, for their applications to real-world networks. For instance, the labeled wiring of these graphs is key to the empirical analysis of metabolic networks, where the metabolism of an organism is studied in terms of the concurrence of smaller functional subnets called modules. We wonder whether our formal connectives may assist in the reconstruction and understanding of the whole metabolism in terms of the concurrence of the modules.

There is much more work to be done still. Both in the applications we are pursuing and on the theory of Dialectica Petri nets. On the theory side notions of behavior (token game) should be investigated and on the practical side we have still to investigate how the implemented systems can be modified to deal with our nets.

Dialectica Petri nets share some of the pros and cons of other Linear Logic based nets. As far as we know no one has investigated Differential Linear Logic Petri nets, yet (see [32] for relating differential interaction nets to the  $\pi$ -calculus). We wonder if this would make the exchange of information with modellers somewhat easier. Finally we would like to investigate whether we could code our nets using Catlab <https://github.com/AlgebraicJulia/Catlab.jl>, a framework for computational category theory, written in the Julia language and already used for other styles of Petri nets.

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## A Omitted proofs

*Proof of Proposition 4.6.* A group is a monoid, thus we only need to check that  $a \multimap b$  actually defines an internal hom and that it respects the ordering.

$$\begin{aligned} b \otimes c &\sqsubseteq a \\ \Leftrightarrow b \otimes c \otimes c^{-1} &\sqsubseteq a \otimes c^{-1} \\ \Leftrightarrow b &\sqsubseteq c \multimap a \end{aligned}$$

$$\begin{aligned} b &\sqsubseteq a \wedge a' \sqsubseteq b' \\ \Rightarrow a^{-1} &\sqsubseteq b^{-1} \wedge a' \sqsubseteq b' \\ \Rightarrow a' \otimes a^{-1} &\sqsubseteq b' \otimes b^{-1} \\ \Rightarrow a \multimap a' &\sqsubseteq b \multimap b' \end{aligned}$$

□

*Proof of Proposition 4.9.* The identity arrow of an object  $U \xleftarrow{\alpha} X$  in  $\mathbf{M}_L\text{Set}$  is given by the pair  $(\mathbb{1}_U, \mathbb{1}_X)$  of identities in  $\text{Set}$ . Moreover, given objects  $A = (U \xleftarrow{\alpha} X)$ ,  $B = (V \xleftarrow{\beta} Y)$ , and  $C = (W \xleftarrow{\gamma} Z)$ , and morphisms  $(f, F): A \rightarrow B$  and  $(g, G): B \rightarrow C$ , their composition is computed componentwise as  $(g, G) \circ (f, F) = (g \circ f, F \circ G): A \rightarrow C$ . Notice that  $(g, G) \circ (f, F)$  is a morphism in  $\mathbf{M}_L\text{Set}$ : given  $u \in U$  and  $z \in Z$ , we have  $\alpha(u, FGz) \sqsubseteq \beta(fu, Gz) \sqsubseteq \gamma(gfu, z)$ . Associativity and unitality come from associativity and unitality in  $\text{Set}$ . □

*Proof of Proposition 4.12.* The object  $A \otimes B = (U \times V \xleftarrow{\alpha \otimes \beta} X^V \times Y^U)$  is clearly an object of  $\mathbf{M}_L\text{Set}$ . The unit is the object  $I = (1 \xleftarrow{e} 1)$ , which assigns to  $1 \times 1$  the monoidal unit  $e$  of  $L$ . On morphisms  $(f, F): A \rightarrow A'$  and  $(g, G): B \rightarrow B'$ , the monoidal product can be defined as

$$(f, F) \otimes (g, G) = (f \times g, F(-)g \times G(-)f): A \otimes B \rightarrow A' \otimes B'$$

where  $f \times g: U \times V \rightarrow U' \times V'$  and  $F(-)g \times G(-)f: X^{V'} \times Y^{U'} \rightarrow X^V \times Y^U$ . We need to check that the monoidal product is well defined, which means that  $(f, F) \otimes (g, G)$  satisfies the condition on morphisms.

$$\begin{aligned} &\alpha \otimes \beta(u, v, (F(-)g \times G(-)f)(f', g')) \\ &= \alpha \otimes \beta(u, v, Ff'g, Gg'f) \\ &= \alpha(u, Ff'Gv) \otimes \beta(v, Gg'fu) \\ &\sqsubseteq \alpha'(fu, f'Gv) \otimes \beta'(gv, g'fu) \\ &= \alpha' \otimes \beta'(fu, gv, f', g') \\ &= \alpha' \otimes \beta'((f \times g)(u, v), f', g') \end{aligned}$$

The monoidal product is a functor as it preserves composition

$$\begin{aligned} &((f', F') \circ (f, F)) \otimes ((g', G') \circ (g, G)) \\ &= (f' \circ f, F' \circ F) \otimes (g' \circ g, G' \circ G) \\ &= ((f' \circ f) \times (g' \circ g), F'F'(-)g'g \times G'G'(-)f'f) \\ &= ((f' \times g') \circ (f \times g), (F'(-)g' \times G'(-)f') \circ (F(-)g \times G(-)f)) \\ &= (f' \times g', F'(-)g' \times G'(-)f') \circ (f \times g, F(-)g \times G(-)f) \\ &= ((f', F') \otimes (g', G')) \circ ((f, F) \otimes (g, G)) \end{aligned}$$

and identities

$$\begin{aligned} &(\mathbb{1}_U, \mathbb{1}_X) \otimes (\mathbb{1}_V, \mathbb{1}_Y) \\ &= (\mathbb{1}_U \times \mathbb{1}_V, \mathbb{1}_X(-)\mathbb{1}_V \times \mathbb{1}_Y(-)\mathbb{1}_U) \\ &= (\mathbb{1}_{U \times V}, \mathbb{1}_{X^V \times Y^U}) \end{aligned}$$

The associator is defined by the following isomorphisms in  $\text{Set}$

$$\alpha_{A,B,C} = (\alpha_{U,V,W}, A_{X,Y,Z}): (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$$

where  $\alpha_{U,V,W}: (U \times V) \times W \rightarrow U \times (V \times W)$  is the associator in  $\text{Set}$  and  $A_{X,Y,Z}: X^{V \times W} \times (Y^W \times Z^V)^U \rightarrow (X^V \times Y^U)^W \times Z^{U \times V}$  is the composition of isomorphisms in  $\text{Set}$  given by

$$\begin{aligned} X^{V \times W} \times (Y^W \times Z^V)^U &\cong X^{V \times W} \times (Y^{U \times W} \times Z^{U \times V}) \\ &\cong (X^{V \times W} \times Y^{U \times W}) \times Z^{U \times V} \cong (X^V \times Y^U)^W \times Z^{U \times V} \end{aligned}$$

The unitors are defined by the following isomorphisms in  $\text{Set}$

$$\lambda_A = (\lambda_U, L_X): I \otimes A \rightarrow A \qquad \rho_A = (\rho_U, R_X): A \otimes I \rightarrow A$$

where  $\lambda_U: 1 \times U \rightarrow U$  and  $\rho_U: U \times 1 \rightarrow U$  are the unitors in  $\text{Set}$ , and  $L_X: X \rightarrow 1^U \times X^1$  and  $R_X: X \rightarrow X^1 \times 1^U$  are the compositions of isomorphisms in  $\text{Set}$  given by

$$X \xrightarrow{\cong} 1 \times X \xrightarrow{\cong} 1^U \times X^1 \qquad X \xrightarrow{\cong} X \times 1 \xrightarrow{\cong} X^1 \times 1^U$$

We are left to prove that the above are actually morphisms in  $\text{M}_L\text{Set}$ , that they are natural isomorphisms and that they satisfy the pentagon and triangle equations [33]. The associator is a morphism because for all  $((u, v), w) \in (U \times V) \times W$  and all  $(f, (g, h)) \in X^{V \times W} \times (Y^W \times Z^V)^U$

$$\begin{aligned} &((\alpha \otimes \beta) \otimes \gamma)((u, v), w), A_{X,Y,Z}(f, (g, h))) \\ &= ((\alpha \otimes \beta) \otimes \gamma)((u, v), w), ((f, g), h)) \\ &= (\alpha(u, f(v, w)) \otimes \beta(v, g(u, w))) \otimes \gamma(w, h(u, v)) \\ &= \alpha(u, f(v, w)) \otimes (\beta(v, g(u, w)) \otimes \gamma(w, h(u, v))) \\ &= (\alpha \otimes (\beta \otimes \gamma))((u, (v, w)), (f, (g, h))) \\ &= (\alpha \otimes (\beta \otimes \gamma))(\alpha_{U,V,W}((u, v), w), (f, (g, h))) \end{aligned}$$

The unitors are morphisms because for all  $u \in U$  and all  $x \in X$

$$\begin{aligned} (I \otimes \alpha)((*, u), L_X(x)) &= (\alpha \otimes I)((u, *), R_X(x)) \\ &= (I \otimes \alpha)((*, u), (*, x)) &= (\alpha \otimes I)((u, *), (x, *)) \\ &= I(*, *) \otimes \alpha(u, x) &= \alpha(u, x) \otimes I(*, *) \\ &= e \otimes \alpha(u, x) &= \alpha(u, x) \otimes e \\ &= \alpha(u, x) &= \alpha(u, x) \end{aligned}$$

The associator and the unitors are natural isomorphisms because they are natural isomorphisms component-wise. Finally, the triangle and pentagon equations hold because they hold in  $\text{Set}$ .  $\square$

*Proof of Proposition 4.14.* The object  $[A, B] = V^U \times X^Y \xrightarrow{[\alpha, \beta]} U \times Y$  is clearly an object of  $\text{M}_L\text{Set}$ . On morphisms  $(f, F): A' \rightarrow A$  and  $(g, G): B \rightarrow B'$  in  $\text{M}_L\text{Set}$ , the internal hom can be defined as

$$[(f, F), (g, G)] = (g(-)f \times F(-)G, f \times G): [A, B] \rightarrow [A', B']$$

where  $g(-)f \times F(-)G: V^U \times X^Y \rightarrow V'^{U'} \times X'^{Y'}$  and  $f \times G: U' \times Y' \rightarrow U \times Y$ . We need to check that the internal hom is well defined, which means that  $[(f, F), (g, G)]$  needs to satisfy the condition on morphisms. For all  $(h, H) \in V^U \times X^Y$  and all  $(u', y') \in U' \times Y'$

$$\begin{aligned} &[\alpha, \beta](h, H, (f \times G)(u', y')) \\ &= [\alpha, \beta](h, H, fu', Gy') \\ &= \alpha(fu', HGy') \multimap \beta(hfu', Gy') \\ &\sqsubseteq \alpha'(u', FHGy') \multimap \beta'(ghfu', y') \\ &= [\alpha', \beta'](ghf, FHG, u', y') \\ &= [\alpha', \beta']((g(-)f \times F(-)G)(h, H), u', y') \end{aligned}$$

because  $\alpha'(u', FHGy') \sqsubseteq \alpha(fu', HGy')$  and  $\beta(hfu', Gy') \sqsubseteq \beta'(ghfu', y')$  as  $(f, F)$  and  $(g, G)$  are morphisms. The internal hom is a functor as it preserves composition

$$\begin{aligned} &[(f', F') \circ (f, F), (g', G') \circ (g, G)] \\ &= [(ff', F'F), (g'g, GG')] \\ &= (g'g(-)ff' \times F'F(-)GG', ff' \times GG') \\ &= ((g'(-)f' \times F'(-)G') \circ (g(-)f \times F(-)G), (f \times G) \circ (f' \times G')) \\ &= (g'(-)f' \times F'(-)G', f' \times G') \circ (g(-)f \times F(-)G, f \times G) \\ &= [(f', F'), (g', G')] \circ [(f, F), (g, G)] \end{aligned}$$

and identities

$$\begin{aligned}
& [\mathbb{1}_A, \mathbb{1}_B] \\
&= [(\mathbb{1}_U, \mathbb{1}_X), (\mathbb{1}_V, \mathbb{1}_Y)] \\
&= (\mathbb{1}_V(-)\mathbb{1}_U \times \mathbb{1}_X(-)\mathbb{1}_Y, \mathbb{1}_U \times \mathbb{1}_Y) \\
&= (\mathbb{1}_{V^U \times X^Y}, \mathbb{1}_{U \times Y}) \\
&= \mathbb{1}_{[A, B]}
\end{aligned}$$

□

*Proof of Theorem 4.15.* To prove the adjunction  $- \otimes B \dashv [B, -]$  we have to show that, for every objects  $A$  and  $C$  in  $\mathbf{M}_L\mathbf{Set}$ , there is a bijection  $\psi_{A,C}: \mathbf{Hom}_{\mathbf{M}_L\mathbf{Set}}(A \otimes B, C) \cong \mathbf{Hom}_{\mathbf{M}_L\mathbf{Set}}(A, [B, C])$  that is natural in  $A$  and  $C$ .

Let  $\phi_{U,Z}: \mathbf{Hom}_{\mathbf{Set}}(U \times V, Z) \rightarrow \mathbf{Hom}_{\mathbf{Set}}(U, Z^V)$  be the natural isomorphism witnessing the adjunction between the cartesian product and the internal hom in  $\mathbf{Set}$  and let  $\sigma_{U,V}: U \times V \rightarrow V \times U$  be the symmetry of the cartesian product in  $\mathbf{Set}$ . Define the maps

$$\begin{aligned}
\psi_{A,C}(f, F) &= (\langle \phi(f), \phi(\phi^{-1}(F_2) \circ \sigma_{U,Z}) \rangle, \phi^{-1}(F_1) \circ \sigma_{V,Z}) \\
\psi_{A,C}^{-1}(g, G) &= (\phi^{-1}(g_1), \langle \phi(G \circ \sigma_{Z,V}), \phi(\phi^{-1}(g_2) \circ \sigma_{Z,U}) \rangle)
\end{aligned}$$

We can check that they are well defined. An element of  $\mathbf{Hom}_{\mathbf{M}_L\mathbf{Set}}(A \otimes B, C)$  is a pair  $(f, \langle F_1, F_2 \rangle)$  with  $f: U \times V \rightarrow W$  and  $F = \langle F_1, F_2 \rangle: Z \rightarrow X^V \times Y^U$  such that  $\forall (u, v) \in U \times V \forall z \in Z (\alpha \otimes \beta)(u, v, Fz) \sqsubseteq \gamma(f(u, v), z)$ , which is equivalent to  $\alpha(u, (F_1(z))(v)) \otimes \beta(v, (F_2(z))(u)) \sqsubseteq \gamma(f(u, v), z)$ . On the other hand, an element of  $\mathbf{Hom}_{\mathbf{M}_L\mathbf{Set}}(A, [B, C])$  is a pair  $(\langle g_1, g_2 \rangle, G)$  with  $g = \langle g_1, g_2 \rangle: U \rightarrow W^V \times Y^Z$  and  $G: V \times Z \rightarrow X$  such that  $\forall u \in U \forall (v, z) \in V \times Z \alpha(u, G(v, z)) \sqsubseteq [\beta, \gamma](g(u), v, z)$ , which is equivalent to  $\alpha(u, G(v, z)) \sqsubseteq \beta(v, (g_2(u))(z)) \multimap \gamma((g_1(u))(v), z)$ .

They are morphisms because the inequality condition for morphisms in  $\mathbf{M}_L\mathbf{Set}$  holds with equality. We check that they are inverses to each other.

$$\begin{aligned}
& \psi_{A,C} \circ \psi_{A,C}^{-1}(g, G) \\
&= (\langle \phi(\phi^{-1}(g_1)), \phi(\phi^{-1}(\phi(\phi^{-1}(g_2) \circ \sigma_{Z,U})) \circ \sigma_{U,Z}) \rangle, \phi^{-1}(\phi(G \circ \sigma_{Z,V})) \circ \sigma_{V,Z}) \\
&= (\langle g_1, g_2 \rangle, G) \\
& \psi_{A,C}^{-1} \circ \psi_{A,C}(f, F) \\
&= (\phi^{-1}(\phi(f)), \langle \phi(\phi^{-1}(F_1) \circ \sigma_{V,Z} \circ \sigma_{Z,V}), \phi(\phi^{-1}(\phi(\phi^{-1}(F_2) \circ \sigma_{U,Z})) \circ \sigma_{Z,U}) \rangle) \\
&= (f, \langle F_1, F_2 \rangle)
\end{aligned}$$

We check that they are natural. Naturality comes from naturality of  $\phi$  in  $\mathbf{Set}$ .

□

*Proof of Lemma 6.1.* The ordering of the poset  $\sqsubseteq$  is given by the usual ordering on the integers,  $a \sqsubseteq b \Leftrightarrow a \leq b$ . The conjunction is associative and unital, with 1 as unit, so it is a monoid operation on  $\mathbb{3}$ . Moreover, it respects the ordering and this makes  $\mathbb{3}$  a partially ordered monoid. We are left to check whether the internal hom is right adjoint to the conjunction, namely, whether  $b \otimes c \leq a \Leftrightarrow b \leq c \multimap a$ . By the definition of internal hom,

$$\begin{aligned}
& b \leq c \multimap a \\
&\Leftrightarrow b \leq \max\{x : x \otimes c \leq a\} \\
&\Leftrightarrow b \otimes c \leq a
\end{aligned}$$

□

*Proof of Lemma 6.3.* The ordering of the poset  $\sqsubseteq$  is given by the opposite of the usual ordering on the natural numbers,  $a \sqsubseteq b \Leftrightarrow a \geq b$ . The conjunction is associative and unital, with 0 as unit, so it is a monoid operation on  $\mathbb{N}$ . Moreover, it respects the ordering and this makes  $\mathbb{N}$  a partially ordered monoid. We are left to check whether the internal hom is

right adjoint to the conjunction, namely, whether  $b \otimes c \sqsubseteq a \Leftrightarrow b \sqsubseteq c \multimap a$ . By the definition of internal hom,

$$\begin{aligned}
& b \sqsubseteq c \multimap a \\
& \Leftrightarrow b \geq \max\{a - c, 0\} \\
& \Leftrightarrow b \geq a - c \wedge b \geq 0 \\
& \Leftrightarrow b + c \geq a \\
& \Leftrightarrow b \otimes c \sqsubseteq a
\end{aligned}$$

□

*Proof of Lemma 6.6.* The image of the product of real numbers in the closed interval  $[0, 1]$  is the closed interval itself. This product is associative and unital, thus it gives a monoid structure to  $[0, 1]$ . Moreover, it preserves the ordering and this makes  $[0, 1]$  a partially ordered monoid. We need to check that truncated division as defined above gives an internal hom.

<p>If <math>c \neq 0 \wedge c \geq a</math></p> $b \sqsubseteq c \multimap a$ $\Leftrightarrow b \leq \frac{a}{c}$ $\Leftrightarrow b \cdot c \leq a$ $\Leftrightarrow b \otimes c \sqsubseteq a$	<p>If <math>c = 0</math></p> $b \sqsubseteq c \multimap a$ $\Leftrightarrow b \leq 1$ $\Leftrightarrow 0 \leq a$ $\Leftrightarrow b \cdot 0 \leq a$ $\Leftrightarrow b \otimes c \sqsubseteq a$	<p>If <math>c &lt; a</math></p> $b \sqsubseteq c \multimap a$ $\Leftrightarrow b \leq 1$ $\Leftrightarrow b \leq 1 < \frac{a}{c}$ $\Leftrightarrow b \cdot c \leq a$ $\Leftrightarrow b \otimes c \sqsubseteq a$
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□

*Proof of Proposition 6.8.*  $(L_1 \times L_2, \otimes, e)$  is the cartesian product of two monoids and therefore it is a monoid.  $(L_1 \times L_2, \leq)$  is a partial ordered set with the ordering defined above. Since  $l_i \leq l'_i$  implies both  $l_i * k_i \leq l'_i * k_i$  and  $k_i * l_i \leq k_i * l'_i$  for each  $k_i \in L_i$  and  $i \in 1, 2$ ; then  $l \leq l'$  implies  $l \otimes k \leq l' \otimes k$  and  $k \otimes l \leq k \otimes l'$  for every  $k = (k_1, k_2) \in L_1 \times L_2$ . This proves that  $(L_1 \times L_2, \leq, \otimes, e, \multimap)$  is a partially ordered monoid. We need to prove that the internal hom defined above is right adjoint to the monoidal product.

$$\begin{aligned}
& b \otimes c \leq a \\
& \Leftrightarrow (b_1 \otimes_1 c_1, b_2 \otimes_2 c_2) \leq (a_1, a_2) \\
& \Leftrightarrow b_1 \otimes_1 c_1 \leq_1 a_1 \wedge b_2 \otimes_2 c_2 \leq_2 a_2 \\
& \Leftrightarrow b_1 \leq_1 c_1 \multimap_1 a_1 \wedge b_2 \leq_2 c_2 \multimap_2 a_2 \\
& \Leftrightarrow (b_1, b_2) \leq (c_1 \multimap_1 a_1, c_2 \multimap_2 a_2) \\
& \Leftrightarrow b \leq c \multimap a
\end{aligned}$$

This proves that the product of lineales is again a lineale.

□