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Natural Number Objects in Dialectica Categories

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Abstract

This note sets down some facts about natural number objects in the Dialectica category $\mathrm{Dial}_2(\mathbf{Sets})$. Natural number objects allow us to model Gödel's *System T* in an intrinsically logical fashion. Gödel's Dialectica Interpretation is a powerful tool originally used to prove the consistency of arithmetic. It was surprising (but pleasing) to discover, in the late eighties, that studying the Dialectica Interpretation by means of categorical proof theory led to models of Girard's Linear Logic, in the shape of Dialectica categories. More recently Dialectica Interpretations of (by now established) Linear Logic systems have been studied, but not extended to *System T*. In this note we set out to to consider notions of natural number objects in the original Dialectica category models of the Interpretation. These should lead to intrinsic notions of linear recursitivity, we hope.

Keywords: categorical logic, linear logic, natural numbers object, dialectica categories

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1 Introduction

This short note describes alternative notions of natural numbers objects in the Dialectica categories. Dialectica categories arose from an internal characterization of Gödel's Dialectica Interpretation, which uses *System T*, a prototypical system for primitive and (generally) recursive functions. There has been much work recently on analyzing the process of computation through the "linear logic perspective". This has produced a body of interesting work investigating concepts such as "linear recursivity", "linear *System T*" and "linear primitive functions" ([1,2]), but we think that more work is needed, if these notions are to truly represent an extension of the Curry-Howard paradigm. The calculations below can be seen as a preparatory steps for the discussion of primitive recursion in general monoidal categories, following the work of [3].

A Natural Numbers Object (or NNO) is an object in a category equipped with structure giving it properties similar to those of the set of natural numbers $\mathbb{N}$ in the category of **Sets**. This means that for each prospective natural numbers object N we need to associate a morphism that plays the role of the constant *zero* in the natural numbers and we need to describe a morphism from N to N that plays the role of the *successor* function. Moreover these morphisms for zero and successor need to help us define *iterators*.

Natural numbers objects have been extensively studied, particularly in the context of toposes, starting with the work of Lawvere ([7]).[5] However, the definition makes sense in any category with finite products, a cartesian category. The definition also makes sense even in categories with less structure than products, i.e. in *monoidal* (closed or not) categories we can define a natural numbers object e.g. [9,8]. Mackie, Román and Abramsky have the following definition:

Definition 1.1 [8] Let $\mathcal{C}$ be a monoidal (closed) category with unit I, a *weak natural numbers object* (NNO) is an object N of $\mathcal{C}$ together with morphisms $\mathrm{zero} : I \longrightarrow N$ and $\mathrm{succ} : N \longrightarrow N$ such that for any object B of $\mathcal{C}$ and morphisms $b : I \longrightarrow B$ and $g : B \longrightarrow B$ there exists a morphism $h : N \longrightarrow B$ such that the diagrams below commute:

$$\begin{tikzcd} I \arrow[r, "\mathrm{zero}"] \arrow[dr, "b"'] & N \arrow[r, "\mathrm{succ}"] \arrow[d, "h"] & N \arrow[d, "h"] \\ & B \arrow[r, "g"'] & B \end{tikzcd}$$

If the morphism $h : N \longrightarrow B$ is *unique* we say that N is a *strong NNO* or simply a NNO. If the morphism is not necessarily unique we talk about a weak natural numbers object.

The paradigmatic example goes as follows: Let $\mathbb{N}$ be the usual natural numbers

[5] Lawvere writes, on introducing an axiom asserting that NNOs exist, "This [axiom] plays the role of our axiom of infinity".

in the category **Sets**. Then 1, the singleton set $*$ and $\mathbb{N}$ together with the usual zero and successor functions (written "ζ" and "$+1$" with $\zeta(*) = 0$ and $+1(n) = n + 1$ for all $n \in \mathbb{N}$) form a natural number structure over 1 in **Sets**. Given $f : 1 \longrightarrow X$ and $g : X \longrightarrow X$ we have the map h given by $h(n) = g^n(f(*))$ makes the following diagram commute and is the unique map doing so.

$$\begin{tikzcd}
1 \arrow[r, "\zeta"] \arrow[dr, "f"'] & \mathbb{N} \arrow[r, "+1"] \arrow[d, "h"'] & \mathbb{N} \arrow[d, "h"'] \\
& X \arrow[r, "g"'] & X
\end{tikzcd}$$

2 Dialectica Categories

Dialectica categories were introduced by de Paiva in her thesis [4]. They were conceived as an internal model of Gödel's Dialectica Interpretation [6], but turned out to be also a model of Linear Logic, then a new logical system introduced by Jean-Yves Girard.

Definition 2.1 Objects of the Dialectica category $\mathrm{Dial}_2(\mathbf{Sets})$ are triples, $A = (U, X, R)$, where U and X are sets and $R \subseteq U \times X$ is a (usual, set-theoretic) relation. Given elements u in U and x in X, either they are related by R, $R(u, x) = 1$ or they are not and $R(u, x) = 0$, hence the 2 in the name of the category.

A morphism from A to $B = (V, Y, S)$ is a pair of functions $f : U \longrightarrow V$ and $F : Y \longrightarrow X$ such that $uRF(y) \Longrightarrow f(u)Sy$. We depict morphisms in this note as

$$\begin{tikzcd}
& X & & Y \arrow[ll, "F"'] \\
U \arrow[ur, no head, "R"] \arrow[rr, "f"'] & & V \arrow[ur, no head, "S"'] &
\end{tikzcd}$$

The category $\mathrm{Dial}_2(\mathbf{Sets})$ has a symmetric monoidal closed structure, which makes it a model of (exponential-free) intuitionistic multiplicative linear logic. We recall the definition of this symmetric monoidal closed structure below.

Definition 2.2 Let $A = (U, X, R)$ and $B = (V, Y, S)$ be objects in $\mathrm{Dial}_2(\mathbf{Sets})$. The *tensor product* of A and B is given by

$$A \otimes B = (U \times V, X^V \times Y^U, R \otimes S)$$

where the relation $R \otimes S$ is given by $(u, v)\ R \otimes S\ (f, g)$ iff $uRf(v)$ and $vSg(u)$. In particular the unit for this tensor product is the object I_{Dial}, $(1, 1, =)$, where $1 = \{*\}$ is a singleton set and $=$ is the identity relation on the singleton set.

The *internal-hom* is given by

$$[A, B] = (V^U \times X^Y, U \times Y, [R, S])$$

where $(f, F)[R, S](u, x)$ iff $uRF(y)$ implies $f(u)Sy$. The tensor product is adjoint to the internal-hom, as usual

$$Hom_{Dial}(A \otimes B, C) \cong Hom_{Dial}(A, [B, C])$$

There is an *auxiliary tensor product* structure given by

$$A \circ B = (U \times V, X \times Y, R \circ S)$$

where $(u, v)R \circ S(x, y)$ iff uRx and vSy. This simpler tensor structure is not the adjoint of the internal-hom. The unit for this tensor product is also I_{Dial}.

The *cartesian product* is given by $A \times B = (U \times V, X + Y, ch)$ where $X + Y = X \times 0 \cup Y \times 1$ and the relation ch (short for 'choose') is given by $(u, v)ch(x, 0)$ if uRx and $(u, v)ch(y, 1)$ if vSy. The unit for this product is $(1, \emptyset, \emptyset)$, the terminal object of $\mathrm{Dial}_2(\mathbf{Sets})$.

The category $\mathrm{Dial}_2(\mathbf{Sets})$ also has *coproducts* given by the dual construction to the one above, namely $A + B = (U + V, X \times Y, ch)$ and an *initial object* $0 = (\emptyset, 1, \emptyset)$.

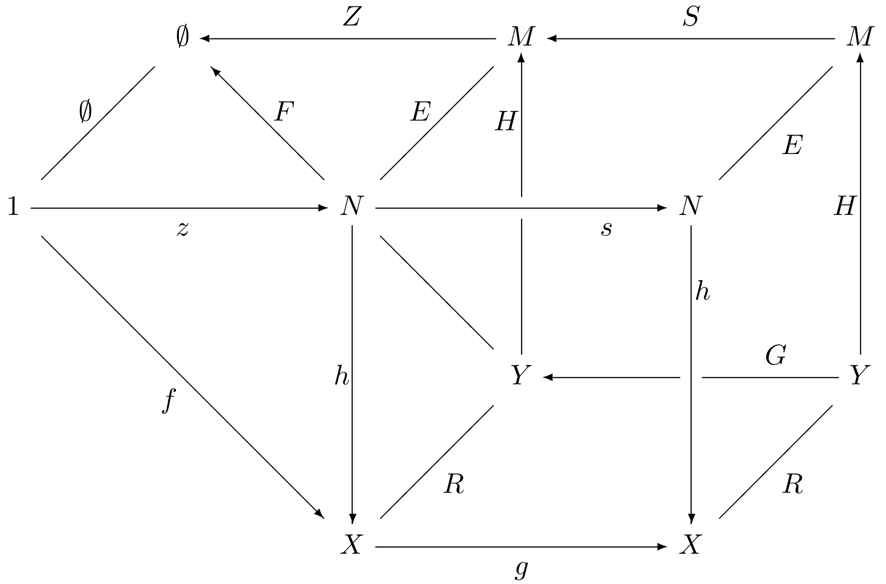
3 Natural Numbers Objects in $\mathrm{Dial}_2(\mathbf{Sets})$

To investigate iteration and recursion in dialectica categories we would like to define a natural numbers object in $\mathrm{Dial}_2(\mathbf{Sets})$. First we need to decide with respect to which one of the monoidal structures in $\mathrm{Dial}_2(\mathbf{Sets})$ we will define our prospective natural numbers object. In principle, we can use either the cartesian structure of $\mathrm{Dial}_2(\mathbf{Sets})$ or any one of its tensor structures.

3.1 Using the cartesian structure

The first candidate monoidal structure is the *cartesian* product in $\mathrm{Dial}_2(\mathbf{Sets})$. This means that we would require a map corresponding to zero from the terminal object $(1, \emptyset, \emptyset)$ in $\mathrm{Dial}_2(\mathbf{Sets})$ to our natural numbers object candidate, say a generic object like (N, M, E).

Reading from Definition (1), (N, M, E) is a NNO with respect to the cartesian structure of $\mathrm{Dial}_2(\mathbf{Sets})$if there are maps $(z, Z) : (1, \emptyset, \emptyset) \longrightarrow (N, M, E)$ and $(s, S) : (N, M, E) \longrightarrow (N, M, E)$ such that for any object (X, Y, R) and any pair of morphisms $(f, F) : (1, \emptyset, \emptyset) \longrightarrow (X, Y, R)$ and $(g, G) : (X, Y, R) \longrightarrow (X, Y, R)$ there exists some (unique) $(h, H) : (N, M, E) \longrightarrow (X, Y, R)$ such that the following diagram, which we refer to below as the 'main diagram,' commutes.



Proposition 3.1 *The category Dial$_2$(**Sets**) has a (trivial) NNO with respect to its cartesian structure, given by* $(\mathbb{N}, \emptyset, \emptyset)$.

Proof. It is clear that demands on the first co-ordinate of a NNO in Dial$_2$(**Sets**) are exactly those as for sets. Consequently, any possible NNO for Dial$_2$(**Sets**) is of the form $N = (\mathbb{N}, M, E)$ for some set M and some relation $E \subseteq \mathbb{N} \times M$, where $\mathbb{N}$ is the usual natural numbers object in **Sets**, with the usual zero constant and the usual successor function on natural numbers.

Since we are using the cartesian structure of Dial$_2$(**Sets**), then there must exist a morphism in Dial$_2$(**Sets**) zero $= (z, Z)\colon 1 \to N$ with two components, $z\colon 1 \to \mathbb{N}$ (as in **Sets**) and $Z\colon M \to 0$. But since the only map into the empty set in the category of **Sets** is the empty map, we would conclude that M is empty and so is E as this is a relation in the product $\mathbb{N} \times \emptyset$.

This trivial NNO works, because given any object B of Dial$_2$(**Sets**) and any maps $f\colon 1 \to B$ and $g\colon B \to B$, we can find a unique map $h\colon N \to B$ making all the necessary NNO diagrams commute. In the first coordinate h is given by the map that exists for $\mathbb{N}$ as a NNO in **Sets** and in the second coordinate this is simply the empty map. $\square$

Note that the existence of the map $1 \to B$ in Dial$_2$(**Sets**) means that B the generic object of the form (X, Y, R) has Y equal to the empty set as its second coordinate and hence R is also the empty relation.

This triviality result is expected, since the 'main' structure of the category Dial$_2$(**Sets**) is the tensor that makes it a symmetric monoidal closed category, not its cartesian structure. This we discuss next.

3.2 Using a monoidal (closed) structure

Tensor products, unlike cartesian products, are not unique up to isomorphism and the category $\mathrm{Dial}_2(\mathbf{Sets})$ has (at least) two prominent tensor products, besides categorical products and coproducts. These tensor products (in Definition 3) share a common unit, the object $(1, 1, =)$, which we could use to obtain a natural number object, as in the monoidal generalization of NNO in Definition 1. Instead of doing this, which in principle would mean investigating *all* possible tensor products in $\mathrm{Dial}_2(\mathbf{Sets})$, we will instead go back to the notion of a *Peano-Lawvere category*, as introduced by Burroni in [3].

Burroni reminds us that the existence of a NNO in a topos $\mathcal{E}$, corresponding to Lawvere's "infinity axiom", only requires the ambient category to possess a terminal object, but the full force of the axiom comes about from the other structure of the topos as well. There are many reasons to formulate a notion of "infinity axiom" in a category where we do not make any other assumptions other than that the ambient is a category E. One of these reasons is that the notions of integer, of recursiviness, of program, of machine, etc.. are notions that one should be able to develop in a uniform way in any mathematical 'universe' E, without assuming other properties of E, because these other properties are not supposed to have any infinitary meaning. Another reason is that there are categories that are very far from being a topos, but that nonetheless satisfy the Peano-Lawvere axiom, or better, a form of this axiom adapted to the absence of hypotheses about the existence of a final object and of cartesian products .

From our part we interested in categories where the main logical structures are monoidal, instead of cartesian, signifying a logic that is resource conscious, but we agree with Burroni that the axiom about infinity should be independent from whether the basic logic is cartesian or monoidal.

Definition 3.2 [3] The axiom of Peano-Lawvere says that for any object X in a category E, there is a diagram of the form

$$X \xrightarrow{\mathrm{zX}} NX \xrightarrow{\mathrm{sX}} NX$$

with the universal property that for any diagram of the form

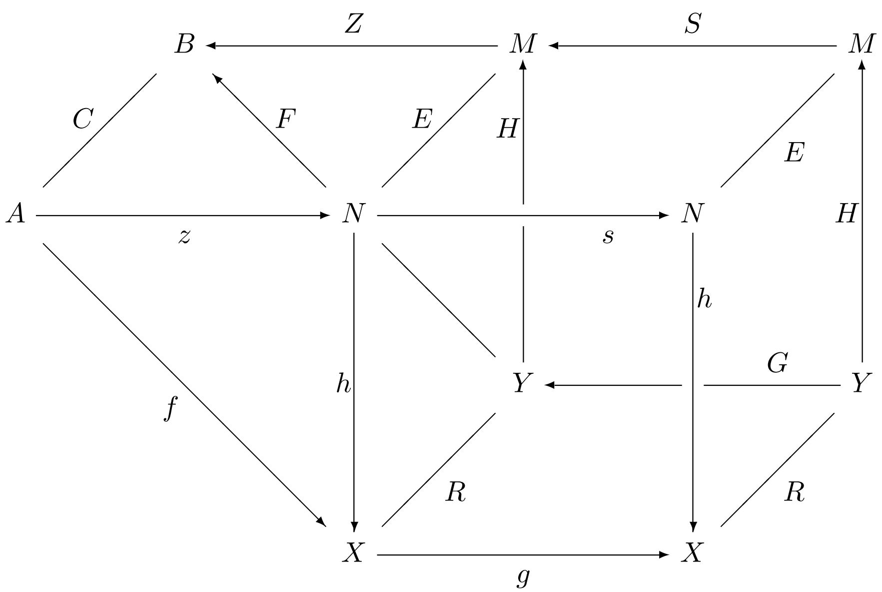
$$X \xrightarrow{\mathrm{f}} Y \xrightarrow{\mathrm{g}} Y$$

there exists $h\colon NX \to Y$ such that the following diagram commutes

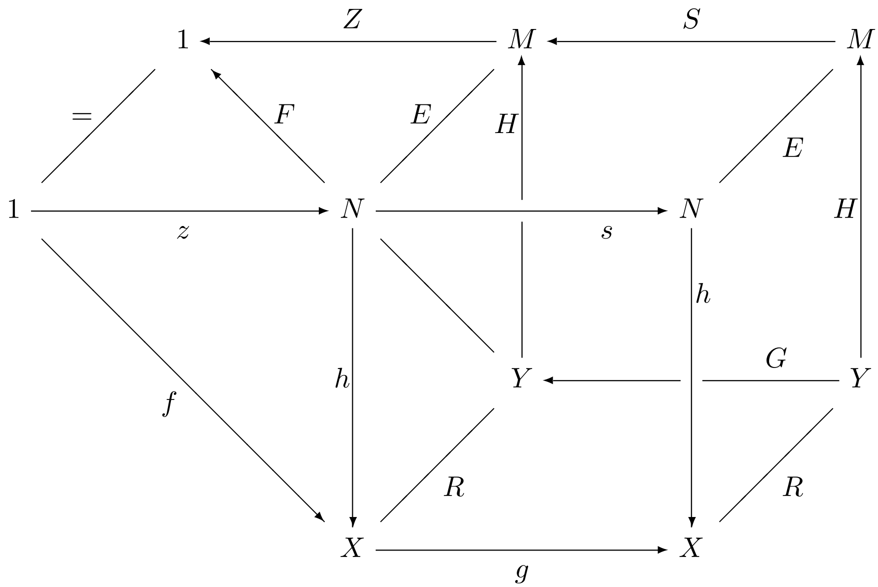
$$\begin{tikzcd}
X \arrow[r, "\mathrm{zX}"] \arrow[dr, "f"'] & NX \arrow[r, "\mathrm{sX}"] \arrow[d, "h"] & NX \arrow[d, "h"] \\
& Y \arrow[r, "g"'] & Y
\end{tikzcd}$$

If a category satisfies this axiom, we say the category is a Peano-Lawvere (PL) category. Is $\mathrm{Dial}_2(\mathbf{Sets})$ a PL-category?

Reading from the definition above, $\mathrm{Dial}_2(\mathbf{Sets})$ is a PL-category if given any object (A, B, C) of $\mathrm{Dial}_2(\mathbf{Sets})$ there is an object of $\mathrm{Dial}_2(\mathbf{Sets})$ (N, M, E) and maps $(z, Z) : (A, B, C) \longrightarrow (N, M, E)$ and $(s, S) : (N, M, E) \longrightarrow (N, M, E)$ such that for any object of $\mathrm{Dial}_2(\mathbf{Sets})$ (X, Y, R) together with a pair of morphisms $(f, F) : (A, B, C) \longrightarrow (X, Y, R)$ and $(g, G) : (X, Y, R) \longrightarrow (X, Y, R)$ there exists some (unique) map in $\mathrm{Dial}_2(\mathbf{Sets})$ $(h, H) : (N, M, E) \longrightarrow (X, Y, R)$ such that the following diagram commutes.



Trying to simplify this picture for the case where the main tensor structure of $\mathrm{Dial}_2(\mathbf{Sets})$ as well as its unit $(1, 1, =)$ are used, we obtain: If $N = (N, M, E)$ is a proposed NNO in $\mathrm{Dial}_2(\mathbf{Sets})$ then there must exist morphisms $zero = (z, Z)\colon (1, 1, =) \to (N, M, E)$ and $succ = (s, S)\colon (N, M, E) \to (N, M, E)$ such that the diagrams below commute.



Proposition 3.3 *The category Dial*$_2$*(*$\mathbf{Sets}$*) has a (trivial) weak NNO with respect to its monoidal closed structure described, given by* $(\mathbb{N}, 1, \mathbb{N} \times 1)$.

Proof. As before, the demands on the first co-ordinate of a NNO in $\mathrm{Dial}_2(\mathbf{Sets})$ are exactly those as for sets. Consequently, any possible NNO for $\mathrm{Dial}_2(\mathbf{Sets})$ is of the form $N = (\mathbb{N}, M, E)$ for some set M and some relation $E \subseteq \mathbb{N} \times M$, where $\mathbb{N}$ is the usual natural numbers object in **Sets**, with the usual zero constant and the usual successor function on natural numbers.

Given that we are using as the unit for the tensor the object $(1, 1 =)$ the morphism *zero* has two components, $z\colon 1 \to \mathbb{N}$ (as in **Sets**) and $Z\colon M \to 1$ on the top of the diagram. The map Z has to be the unique map $!_M\colon M \to 1$ sending all m's in M to the singleton set $*$, as this is what it means to say that 1 is the terminal object in **Sets**. The morphism $succ\colon N \to N$ in $\mathrm{Dial}_2(\mathbf{Sets})$ also has two components (s, S), where $s\colon \mathbb{N} \to \mathbb{N}$ is the usual successor function in $\mathbb{N}$, and $S\colon M \to M$ is to be determined, satisfying some equations.

We need to consider the objects B of $\mathrm{Dial}_2(\mathbf{Sets})$ for which there are maps $(f, F)\colon I \to B$ and $(g, G)\colon B \to B$. Every object B in $\mathrm{Dial}_2(\mathbf{Sets})$ has at least one map to itself, namely the identity, but not every object in $\mathrm{Dial}_2(\mathbf{Sets})$ has a map $(f, F)\colon I \to B$.

Fact 1. If there is a map $(f, F)\colon I \to B$ in $\mathrm{Dial}_2(\mathbf{Sets})$ for a generic object B of the form (X, Y, R) then there exists x_0 in X such for all y in Y we have $x_0 R y$.

Proof. By definition of maps in $\mathrm{Dial}_2(\mathbf{Sets})$, we must have

$$\begin{tikzcd} 1 \arrow[d, "f"'] & 1 \arrow[l, "="'] \\ X & Y \arrow[l, "R"] \arrow[u, "F =!"'] \end{tikzcd}$$

where the map in the left f simply picks up an element of X and the map on the right F is the unique terminal map, hence $\forall * \in 1, \forall y \in Y$, $* = *$ implies $x_0 = f(*)Ry$. □

Fact 2. If there is a NNO in $\mathrm{Dial}_2(\mathbf{Sets})$ of the form $(\mathbb{N}, M, E)$, where M is the singleton set 1, then $S\colon 1 \to 1$ is the identity on 1 and E relates every n in $\mathbb{N}$ to $*$.

Proof. If $N = (\mathbb{N}, 1, E)$ is a NNO in $\mathrm{Dial}_2(\mathbf{Sets})$ then the map $zero = (z, Z)\colon I \to N$ has to be the zero map in $\mathbb{N}$ together with the terminal map in 1 and the $succ = (s, S)\colon N \to N$ consists of the usual successor function on the integers and $S\colon 1 \to 1$ has to be the identity on 1.

The fact that (z, Z) is a map of $\mathrm{Dial}_2(\mathbf{Sets})$ gives us the diagram

$$\begin{tikzcd} 1 \arrow[d, "0"'] & 1 \arrow[l, "="'] \\ \mathbb{N} & 1 \arrow[l, "E"] \arrow[u, "Z = id_1"'] \end{tikzcd}$$

and the condition says for all $*$ in 1, if $* = Z(*)$ then $0(*)E*$. Hence E must be such that $0E*$.

The fact that (s, S) is map of $\mathrm{Dial}_2(\mathbf{Sets})$ gives us the diagram

$$\begin{tikzcd} \mathbb{N} \arrow[d, "s"'] & 1 \arrow[l, "E"'] \\ \mathbb{N} & 1 \arrow[l, "E"] \arrow[u, "S"'] \end{tikzcd}$$

and the condition on morphisms says for all n in $\mathbb{N}$ and for all $*$ in 1, if $nES*$ then $n + 1 = s(n)E*$. But S is the identity on 1, ie $S* = *$, so if $nES* \Rightarrow n + 1E*$, which is just what we need to prove that E relates every n in $\mathbb{N}$ to $*$. □

Back to the proof of the proposition we now have:

The object of $\mathrm{Dial}_2(\mathbf{Sets})$ of the form $(\mathbb{N}, 1, E)$, where E relates every n in $\mathbb{N}$ to $*$, together with morphisms $zero = (0, id_1): I \to N$ and $succ = (+1, id_1): N \to N$ is a weak NNO in $\mathrm{Dial}_2(\mathbf{Sets})$.

Let B be an object (X, Y, R) of $\mathrm{Dial}_2(\mathbf{Sets})$ such that there are maps $(f, F): I \to B$ and $(g, G): B \to B$. To prove $N = (\mathbb{N}, 1, E)$, where $nE*$ for all n in $\mathbb{N}$ is a weak NNO, we must be able to define a map $(h, H): N \to B$ such that the main NNO diagram commutes.

It is clear that $h: \mathbb{N} \to X$ can be defined using the fact that $\mathbb{N}$ is a NNO in **Sets**. It is clear that we must take $H: Y \to 1$ as the terminal map on Y. We need to check that all the required conditions are satisfied.

The required conditions amount to showing that

(i) the map (h, H) is a map of $\mathrm{Dial}_2(\mathbf{Sets})$;

(ii) the triangle commutes, and

(iii) the square commutes in the diagram that we repeat again below to facilitate the reading of this note.

$$
\begin{tikzcd}
 & 1 & & 1 \arrow[ll, "Z = id_1"'] & & 1 \arrow[ll, "S = id_1"'] \\
1 \arrow[ur, dash, "="] \arrow[rr, "z = 0"'] \arrow[ddrr, "f"'] & & \mathbb{N} \arrow[ul, "F = !_Y"'] \arrow[ur, dash, "E"] \arrow[rr, "s = +1"'] \arrow[dr, dash] \arrow[dd, "h"'] & & \mathbb{N} \arrow[ur, dash, "E"'] \arrow[dd, "h"] & \\
 & & & Y \arrow[uu, "H"] & & Y \arrow[ll, "G"'] \arrow[uu, "H = !_Y"'] \\
 & & X \arrow[ur, dash, "R"'] \arrow[rr, "g"'] & & X \arrow[ur, dash, "R"'] &
\end{tikzcd}
$$

We deal with item (i) last as it is more involved.

Commutativity of the triangle diagram (ii) in $\mathrm{Dial}_2(\mathbf{Sets})$ is easy. Note that the diagram in $\mathrm{Dial}_2(\mathbf{Sets})$

$$
\begin{tikzcd}
1 \arrow[r, "zero"] \arrow[dr, "f"'] & N \arrow[d, "h"] \\
 & B
\end{tikzcd}
$$

corresponds to two triangles in **Sets** and in our main diagram:

$$
\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "f"'] & \mathbb{N} \arrow[d, "h"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & 1 \arrow[l, "id_1"'] \\
 & Y \arrow[ul, "F =!"] \arrow[u, "H =!"']
\end{tikzcd}
$$

The left triangle is satisfied because $\mathbb{N}$ is a NNO in **Sets** and the right triangle is trivially satisfied, because 1 is a terminal object.

The Dial_2(**Sets**) condition on morphisms is also satisfied:

$$
\begin{tikzcd}
1 \arrow[d, "0"'] & 1 \arrow[l, "="'] \\
\mathbb{N} \arrow[d, "h"'] & 1 \arrow[l, "E"] \arrow[u, "id_1"'] \\
X & Y \arrow[l, "R"] \arrow[u, "!"']
\end{tikzcd}
$$

The first square says that for all $* \in 1, * \in 1$, if $* = *$ then $0(*)E*$ or $0E*$ which is true. The second square says for all $n \in \mathbb{N}$, for all $y \in Y$ if $nE*(y)$ then $h(n)Ry$, the condition on $(h, !)$ being a map in Dial_2(**Sets**). If both squares commute then the rectangle says for all $*$ in 1 and for all $y \in Y$, if $* = *$ then $f(*)Ry$, which we know.

The square relating the successor function to the function defined by iteration h (item (iii)) commutes.

$$
\begin{tikzcd}
N \arrow[r, "succ"] \arrow[d, "h"'] & N \arrow[d, "h"] \\
B \arrow[r, "g"'] & B
\end{tikzcd}
$$

This corresponds to the squares:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "+1"] \arrow[d, "h"] & \mathbb{N} \arrow[d, "h"] \\ X \arrow[r, "g"'] & X \end{tikzcd} \qquad \begin{tikzcd} 1 & 1 \arrow[l, "S = id_1"'] \\ Y \arrow[u, "!"] & Y \arrow[l, "G"] \arrow[u, "!"'] \end{tikzcd}$$

As before the left square is true by definition of NNO, we choose h so that this commutes and the right square commutes because we are using the terminal map.

Now to show that the proposed map (h, H) is a map in $\mathrm{Dial}_2(\mathbf{Sets})$ we have to work a little. The map (h, H) is a map in $\mathrm{Dial}_2(\mathbf{Sets})$ if the condition for all m in $\mathbb{N}$, for all y in Y, if $mEH(y)$ then $h(m)Ry$ is satisfied. Since $H(y) = *$ and we know $mE*$ for all m in $\mathbb{N}$, we need to show $h(m)Ry$ for all $m \in \mathbb{N}$ and all y in Y.

If $m = 0$ we need to show $h(0)Ry$ for all $y \in Y$. But since $\mathbb{N}$ is the NNO in **Sets** we know that

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "f"'] & \mathbb{N} \arrow[r, "+1"] \arrow[d, "h"] & \mathbb{N} \arrow[d, "h"] \\ & X \arrow[r, "g"'] & X \end{tikzcd}$$

commutes, hence $h(0) = f(*)$ and $h(m+1) = g(h(m))$. But since $B = (X, Y, R)$ is an object that has a map $(f, F)\colon I \to B$ we know (Fact 1) that there exists $x_0 = f(*)$ such that $f(*)Ry$ for all y in Y and hence $h(0) = f(*)Ry$ for all $y \in Y$.

If m is not zero, then $m = n + 1$ and $h(n+1) = g(h(n))$ by the definition of h in **Sets**. But B is an object of $\mathrm{Dial}_2(\mathbf{Sets})$ equipped with a map $(g, G)\colon B \to B$, which means that there exist $g\colon X \to X$ and $G\colon Y \to Y$ in **Sets** such that for all $x \in X$ and for all $y \in Y$, if $xRG(y)$ then $g(x)Ry$. To show that $h(n)Ry$, since we know that $h(0)Ry$ we need to show that if $h(n)Ry$ for all $y \in Y$ then $h(n+1)Ry$ for all $y \in Y$.

But if $h(n)Ry$ for all $y \in Y$, then in particular $h(n)RG(y')$ for all y's that happen to be in the range of G, that is if y happens to be Gy'. In this case $g(h(n))Ry'$, that is $h(n+1)Ry'$. $\square$

Summing up: We obtain a degenerate weak NNO, where in the first coordinate we have business as usual in **Sets** and in the second coordinate we have simply the singleton set 1 and terminal maps.

4 Conclusions

We expected to find a NNO in the dialectica categories, with iteration and recursion as usual in the first coordinate, but co-recursion/co-iteration in the second coordinate, following the pattern in dialectica categories of business as usual in the first coordinate and the dual case of the usual in the second coordinate. It is disappointing to obtain only a 'degenerate' NNO as above, where the second coordinate is trivial. Maybe we have not got the right level of generality.

A remark on related work: Dialectica objects are similar to Chu spaces, which in turn are very similar to Vicker's *topological systems*[11]. But morphisms are very different and as a result, the structure of the categories is fairly different too. Some comparisons are drawn in [5].

It is well known (and a nice description can be found in [10]) that natural number algebras $1 \to N \leftarrow N$ are in bijective correspondence with F-algebras where F is the endofunctor $F(X) = 1 + X$ and that the initial algebra for this functor in **Set** is indeed the usual natural numbers, where we have an isomoprhism $N \cong 1 + N$. Since this is an isomorphism we could also see it as an F-coalgebra, but this is not final in the category of sets. As Plotkin remarks this coalgebra is final in the category of sets and partial functions **Pfn**. Can we change our working underlying category of $\mathrm{Dial}_2(\mathbf{Sets})$ so that a non-trivial NNO can be constructed? Co-recursion is not as well-understood as recursion, in particular we know of no work on co-induction in a linear (or monoidal) situation. More work seems required.

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