Constructive Temporal Logic, Categorically

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Preface

This paper is dedicated to the memory of Professor Grigori Mints, to whom the first author owes a huge debt. Not only an intellectual debt (most people working on proof theory nowadays owe this one), not only a friendship one (there are plenty of us who owe this), but also a mentoring (and a personal help when I needed it) debt. Grisha would not be gratuitously conversational, you could say that his style was 'tough love': work hard, then he would talk to you. But when you needed him, he was there for you. This work might not be, yet, at the stage that he would approve of it, especially given all his work on Dynamic Topological Logic with Kremer and others [17], which might be related to what we describe here. However, this is the best that we can do in the time we have, so it will have to do.

The second author does owe Grigori an intellectual debt. While I never got to meet him in person I do remember fondly reading his "A Short Introduction to Intuitionistic Logic" [20]. His book gave me several realizations about intuitionistic logic that I had previously lacked. It was his book that turned on the light, and I thank him for that.

We would like to thank the anonymous reviewers and Jane Spurr for dealing with a last minute, but heartfelt, submission to this volume.

1 Motivation

Generally speaking, Temporal Logic is any system of rules and symbolism for representing, and reasoning about propositions qualified in terms of time. Temporal logic is one of the most traditional kinds of modal logic, introduced by Arthur Prior in the late 1950s, but it is also one of the most controversial kinds of modal logic, as people have different intuitions about time, how to represent it, and how to reason about it.

There has been a large amount of work in Modal Logic in the last sixty years, mainly in classical modal logic. We are mostly interested in constructive systems, not classical ones. In particular we are interested in a constructive version of temporal logic that satisfies some well-known and desirable proof-theoretical properties, but that is also algebraically and category-theoretically *well-behaved*.

Prior's 'Time and Modality' [24] introduced a propositional modal logic with two temporal connectives (modal operators), F and P, corresponding to "sometime in the Future" and "sometime in the Past". This propositional system has been called **tense logic** to distinguish it from other temporal systems.

Ewald [10] produced a first version of an intuitionistically based temporal logic system with not only operators for "sometime in the Future" and "sometime in the Past", but also operators for "in all future times" and "for all past times". The intuitive reading of these operators is very reasonable:

- P "It has at some past time been the case that"
- F "It will at some future time be the case that"
- H "It has always been the case in the past that"
- G "It will always be the case in the future that"

Ewald and most of the researchers that followed his path of constructivization of tense logic, did so assuming a symmetry between past and future. This symmetry, as well as the symmetry between universal and existential quantifiers, both in the past and in the future, are somewhat at odds with intuitionistic reasoning. In particular while an axiom like $A \to GPA$ "What is, will always have been" makes sense in a constructive way of thinking, the dual one $A \to HFA$ paraphrased in the Stanford Encyclopedia of Philosophy as "What is, has always been going to be" feels very classical.

Constructivizing a classical system is alwys prone to proliferation of systems, as is evident when considering the several versions of intuitionistic set theory, for

$$
\begin{array}{ccc}
\Diamond(A \vee B) & \to & \Diamond A \vee \Diamond B \\
\Diamond\bot & \to & \bot
\end{array}
$$

Figure 1: Distributivity rules

example. In particular the basic constructive modal logic S4 (using Lewis' original naming convention) has two main variants.

The first version of an intuitionistic S4, originally presented by Dag Prawitz in his Natural Deduction book [23] does not satisfy the distributivity of the possibility operator $\Diamond$ over the logical disjunction. Prawitz's system satisfies neither the binary distribution nor its nullary form, as given in Figure 1. We call this system $\mathsf{CS4}$. This system was investigated from a proof theoretical and categorical perspective in [5].

The second main version of an intuitionistic modal S4 does enforce these distributivities and it was thoroughly investigated in Simpson's doctoral thesis [25]. This system is part of a framework for constructive modal logics, based on incorporating, as part of the syntax, the intended semantics of the modal logic, as possible worlds. We call this system $\mathsf{IS4}$.

Ewald's tense logic system consists of a pair of Simpson-style S4 operators [25], representing past and future over intuituionistic propositional logic. This is historically inaccurate, as Simpson based his systems in Ewald's, but it will serve to make some of our main points clearer below. The system we describe in this note is the tense logic system obtained by joining together two pairs of Prawitz-style S4 operators. So it satisfies some of Ewald's rules, but not all.

Simpson remarks that intuitionistic or constructive modal logic is full of interesting questions. As he says:

> Although much work has been done in the field, there is as yet no consensus on the correct viewpoint for considering intuitionistic modal logic. In particular, there is no single semantic framework rivalling that of possible world semantics for classical modal logic. Indeed, there is not even any general agreement on what the **intuitionistic analogue** of the basic modal logic, K, is.

In an intuitionistic logic we do not expect perfect duality between quantifiers, ($\forall x.P(x)$ is not the same as $\neg\exists x.\neg P(x)$) or even between conjunction and disjunction (full De Morgan laws do not hold for intuitionistic propositional logic). So one should not expect a perfect duality between intuitionistic possibility and necessity either. But considerations from first principles do not seem to indicate clearly whether distributivity rules as the ones in Figure 1 should hold or not. Hence it seems sensible to develop different kinds of systems in parallel, proving equivalences, whenever possible. In this paper we develop the idea of tense logic in Prawitz' style. We recall some deductive systems for this tense logic and provide categorical semantics for it.

Much has been done recently in the proof theory of constructive modal logics using more informative sequent systems (e.g. hypersequents, labelled sequents, nested sequents, tree-style sequents, etc..) In particular nested sequents have been used to produce 'modal cubes' for the two variants of constructive modal logics described above. See the pictures below from [2, 26].

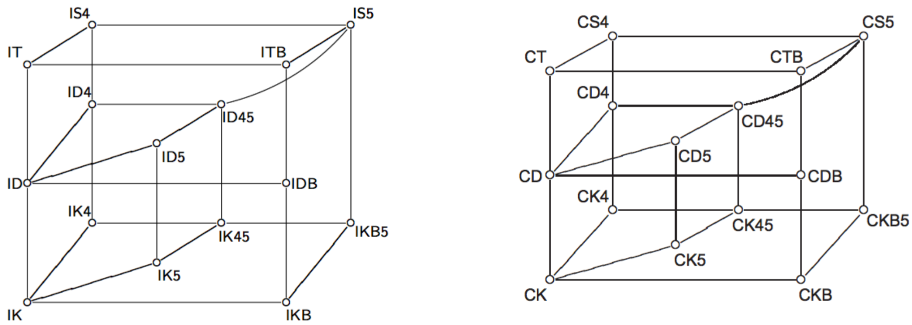

Figure 2: Intuitionistic and Constructive Modal Cubes

Sequent calculi by themselves are not enough to provide us with Curry-Howard correspondences and/or term assignments for these systems. However, using the Prawitz S4 version of these modal systems we can easily produce a Curry-Howard correspondence and a categorical model for the Prawitz-style intuitionistic tense logic, our goal in this paper.

We start by recalling the system using axioms, plain sequent calculus and plain natural deduction. In the next section we describe a term assignment based on the dual calculus described in [12] and show some of its syntactic prooperties. The next section introduces the categorical model (a cartesian closed category with two intertwinned adjunctions) and show the usual soundness and completeness results. Finally we discuss potential applications and limitations of our constructive tense logic.

$$\frac{}{\Delta, A \vdash A}\ \mathrm{Id} \qquad \frac{\Gamma \vdash B \quad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \mathrm{Cut} \qquad \frac{}{\Gamma, \perp \vdash A}\ \perp_{\mathcal{L}}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee_{\mathcal{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{\mathcal{R}_1} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{\mathcal{R}_2}$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\mathcal{L}_1} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\mathcal{L}_2} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_{\mathcal{R}}$$

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ \to_{\mathcal{L}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to_{\mathcal{R}}$$

Figure 3: Intuitionistic Propositional Calculus (LJ)

2 Tense Logic CS4-style

We build up to the constructive tense logic we are interested in TCS4 in progressive steps. We start with the intuitionistic basis LJ, add the modalities to get the constructive modal S4 system, CS4, provide the dual context modification (to help with the reuse of libraries, amongst other things), obtaining dual CS4, DCS4 and then finally consider the two adjunctions to obtain the tense constructive system TCS4.

2.1 Intuitionistic Sequent Calculus

We start by recalling the basic sequent calculus for intuitionistic propositional logic, Gentzen's intuitionistic sequent calculus LJ. The syntax of formulas for LJ is defined by the following grammar:

$$A \quad ::= \quad p \,|\perp|\, A \wedge A \mid A \vee A \mid A \to B$$

The formula p is taken from a set of countably many propositional atoms. The constant $\top$ could be added, but it is the negation of the the falsum constant $\perp$. The initial inference rules, which just model propositional intuitionistic logic, are as in Figure 3.

Sequents denoted $\Gamma \vdash C$ consist of a multiset of formulas, (written as either Γ, Δ, or a numbered version of either), and a formula C. The intuitive meaning is that the conjunction of the formulas in Γ entails the formula C. So far this is our intuitionistic basis.

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\ \Box_{\mathcal{L}} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma, \Delta \vdash \Box A}\ \Box_{\mathcal{R}}$$

$$\frac{\Box\Gamma, A \vdash \Diamond B}{\Delta, \Box\Gamma, \Diamond A \vdash \Diamond B}\ \Diamond_{\mathcal{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{R}}$$

Figure 4: Constructive S4 modal rules (CS4)

2.2 Constructive modal S4

Next we recall the sequent calculus formalization of system CS4 as described in [5].

We recap the modality rules in Figure 4. These, in addition to the initial set of inference rules, define the sequent calculus for CS4. In Figure 3, we write $\Box\Gamma$ for the sequence of boxed formulas $\Box G_1, \Box G_2, \ldots, \Box G_k$ where Γ is the set $G_1, G_2, \ldots, G_k$.

Note that we do have right rules and left rules for introducing the new modal operators $\Box$ (necessity) and $\Diamond$ (posssibility), but these rules are not as symmetric as the propositional ones. Most importantly, we have a local restriciton on the rule that introduces the $\Box$ operator: We can only introduce $\Box$ in the conclusion, if all the assumptions are already boxed. Also the rules for the $\Diamond$ operator presuppose that you have already defined $\Box$ operators. This system is indeed constructive, $\Box$ and $\Diamond$ are independent logical operators and $\Box A$ is not logically equivalent to $\neg\Diamond\neg A$, nor is $\Diamond A$ logically equivalent to $\neg \Box \neg A$. Note that the necessity only fragment is well-behaved and closed, while to define the possibility operator you need a necessity operator in place.

This system has a reasonably nice proof theory. Bierman and de Paiva [5] show that it has a Hilbert-style presentation, a Natural Deduction presentation, as well as a sequent calculus presentation and these presentations are provably equivalent, that is, they prove the same theorems. The sequent calculus satisfies cut-elimination, an old result from Ohnishi and Matsumoto [21], as well as a form of the subformula property. The Natural Deduction formulation has a colourful history: one of its distinct features is that it was described in Prawitz' seminal book in Natural Deduction [23], hence it is sometimes called Prawitz' S4 intuitionistic modal logic. Most interestingly the system has both Kripke and categorical semantics, described respectively in [1] and [5] as well as an independent mathematical semantics in terms of simplicial sets, described by Goubault-Larrecq [14].

$$\frac{\Gamma; \emptyset \vdash A}{\Gamma; \Delta \vdash \Box A}\ \Box_{\mathcal{I}} \qquad \frac{\Gamma; \Delta \vdash \Box A \quad \Gamma, A; \Delta \vdash B}{\Gamma; \Delta \vdash B}\ \Box_{\mathcal{E}}$$

$$\frac{\Gamma; \Delta \vdash A}{\Gamma; \Delta \vdash \Diamond A}\ \Diamond_{\mathcal{I}} \qquad \frac{\Gamma; \Delta \vdash \Diamond A \quad \Gamma; A \vdash \Diamond B}{\Gamma; \Delta \vdash \Diamond B}\ \Diamond_{\mathcal{E}}$$

Figure 5: The dual context modal calculus (DCS4)

2.3 The dual context modal S4 calculus

An equivalent (in terms of provability) but more type-theoretic system can be produced for the modal logic CS4. This is not so well-known, but this system can be given a presentation in terms of a categorical adjunction, between two cartesian closed categories, as we will show in the next section. This categorical presentation has been described both in [5] and in [12], in the former, this is called the **multi-context** formulation of CS4 and the rules are given in Figure 5. (We prefer to call it the dual context sequent calculus.) Note that the rules are Natural Deduction rules, as it should be clear from the fact that they are introduction and elimination rules.

The main difference between the system CS4 and the dual context formulation of CS4 is the fact that the context now has modal formulas and non-necessarily modal ones, separated by a semi-colon as in $\Gamma; \Delta$. The previously difficult rule of $\Box$ introduction now says that to introduce a necessity operator $\Box$ on a conclusion, we need to have an empty context of non-modal assumptions (that is, all the assumptions of this conclusion must be modal). This corresponds to the traditional idea that to prove something is necessarily the case, all its assumptions have to be also necessary (or it must have no assumptions whatsoever).

These rules have been shown by Benton [4] and Barber [3] to correspond to an adjunction of the categories, in the case where the basis is Linear Logic and the modalities correspond to the exponentials. Instead of Linear Logic, we deal with constructive modal logic and the adjunction is between functors corresponding to operators $\Diamond \vdash \Box$.

2.4 The tense CS4 calculus

Finally to get to the tense logic which is the main aim of this note, we need two such adjunctions, but intertwined. This follows the pattern explained by Ewald [10]. Thus $\Diamond$ is left-adjoint to $\blacksquare$ and $\blacklozenge$ is left-adjoint to $\Box$, where we are writing $\blacksquare$ for the operator we called past universal H before and $\Box$ for the future necessity operator

G. The past existential P is $\Diamond$ and the future existential is $\blacklozenge$ or F.

A sequent calculus system for this constructive tense logic is given by the rules in Figure 6. This can be transformed into Natural Deduction in the style of [5] as shown in Figure 8. The problem is that the last two adjunction rules in Figure 6 (that relate the two sets of modalities) are extremely badly-behaved proof-theoretically (no cut-elimination and no subformula property even for cut-free proofs), as discussed in page 35 of Benton's full report [4]. In fact they are the reason for moving to a dual context calculus, as explained in that paper and also in Barber's work [3].

The dual context systems, as described in Barber and Benton's work, are proved equivalent to the system with a single modality operator, either '!' or '$\Box$'. This is because in Intuitionistic Linear Logic one is not usually interested in either *why not?* '?' or '$\Diamond$'. (In Classical Linear Logic the possibility modality is defined by negation of the necessity modality, so this extension is easier to make [22].) Given that our main goal is to discuss categorical semantics, which we can do easily for the necessity modalities, in this note we consider only two necessity-like modalities $\Box$ and $\blacksquare$.

We would like to have a natural deduction version of the tense calculus in dual context style. A dual context-style presentation of a single necessity modality has been presented in Figure 5. Now we need to add another necessity-like modality and discuss their interaction. A preliminary attempt at such calculus is given in Figure 9.

This corresponds to an intuitionistic tense logic obtained by extending IPL with two pairs of adjoint modalities $(\blacklozenge, \Box)$ and $(\Diamond, \blacksquare)$, with no explicit relationship between the modalities of the same colour, namely, $(\blacklozenge, \blacksquare)$ and $(\Diamond, \Box)$.

2.5 Axioms

Axiom sets for the system $\mathsf{TCS4}$ are easier to provide. We need a set for the basic system intuitionistic logic LJ, and any traditional set would do, plus the axioms for modalities, as well as the rules **modus ponens** and **necessitation** for the two necessity operators:

We have similar axioms to Ewald's [10], except that the duality between necessity and possibility is not strict (Ewald's original axioms (7) and (7') in page 171 of [10] are not valid) and that the possibility modalities we deal with, do not distribute over disjunction (Ewald's axioms (4) and (4') are not valid). Also note we do have introspection and reflexivity valid, which correspond to Ewald's extra axioms (12) and (12'), as well as (13) and (13').

We are interested in the term assignment system and its properties, as our aim is to use these as type systems for innovative programming languages. So we needed to provide the systematic work that shows basic properties of the type system $\mathsf{TCS4}$

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\ \Box_{\mathcal{L}} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma, \Delta \vdash \Box A}\ \Box_{\mathcal{R}}$$

$$\frac{\Box\Gamma, A \vdash \blacklozenge B}{\Delta, \Box\Gamma, \blacklozenge A \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{R}}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \blacksquare A \vdash B}\ \blacksquare_{\mathcal{L}} \qquad \frac{\blacksquare\Gamma \vdash A}{\blacksquare\Gamma, \Delta \vdash \blacksquare A}\ \blacksquare_{\mathcal{R}}$$

$$\frac{\blacksquare\Gamma, A \vdash \Diamond B}{\Delta, \blacksquare\Gamma, \Diamond A \vdash \Diamond B}\ \Diamond_{\mathcal{L}} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{R}}$$

$$\frac{A \vdash \Box B}{\blacklozenge A \vdash B}\ adj1_{\mathcal{L}} \qquad \frac{\blacklozenge A \vdash B}{A \vdash \Box B}\ adj1_{\mathcal{R}}$$

$$\frac{A \vdash \blacksquare B}{\Diamond A \vdash B}\ adj2_{\mathcal{L}} \qquad \frac{\Diamond A \vdash B}{A \vdash \blacksquare B}\ adj2_{\mathcal{R}}$$

Figure 6: Tense S4 sequent rules (biCS4)

$$\frac{\Theta \vdash_{\mathcal{C}} X \quad Y, \Phi \vdash_{\mathcal{C}} Z}{\Theta, X \to Y, \Phi \vdash_{\mathcal{C}} Z} \to_l^{\mathcal{C}} \qquad \frac{\Theta, X \vdash_{\mathcal{C}} Y}{\Theta \vdash_{\mathcal{C}} X \to Y} \to_r^{\mathcal{C}} \qquad \frac{\Theta; \emptyset \vdash_{\mathcal{L}} A}{\Theta \vdash_{\mathcal{C}} \mathsf{G}\, A} \mathsf{G}_r$$

$$\frac{\Theta \vdash_{\mathcal{C}} X \quad Y, \Phi; \Gamma \vdash_{\mathcal{L}} A}{\Theta, X \to Y, \Phi; \Gamma \vdash_{\mathcal{L}} A} \to_l^{\mathcal{L}} \qquad \frac{\Theta; \Gamma, A \vdash_{\mathcal{L}} B}{\Theta; \Gamma \vdash_{\mathcal{L}} A \multimap B} \multimap_r^{\mathcal{L}}$$

$$\frac{\Theta; \Gamma \vdash_{\mathcal{L}} A \quad \Phi; \Delta, B \vdash_{\mathcal{L}} C}{\Theta; \Gamma, A \multimap B, \Delta \vdash_{\mathcal{L}} C} \multimap_l^{\mathcal{L}} \qquad \frac{\Theta \vdash_{\mathcal{C}} X}{\Theta; \emptyset \vdash_{\mathcal{L}} \mathsf{F}\, X} \mathsf{F}_r \qquad \frac{\Theta, X; \Gamma \vdash_{\mathcal{L}} A}{\Theta; \mathsf{F}\, X, \Gamma \vdash_{\mathcal{L}} A} \mathsf{F}_l$$

$$\frac{\Theta; B, \Gamma \vdash_{\mathcal{L}} A}{\Theta, \mathsf{G}\, B; \Gamma \vdash_{\mathcal{L}} A} \mathsf{G}_l$$

Figure 7: LNL sequent rules

$$\frac{\Gamma \vdash \Box B}{\Gamma \vdash B}\ \Box_{\mathcal{E}} \qquad \frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k \quad \Box A_1, \ldots \Box A_k \vdash B}{\Gamma \vdash \Box B}\ \Box_{\mathcal{I}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{I}}$$

$$\frac{\Gamma \vdash \blacksquare A}{\Gamma \vdash A}\ \blacksquare_{\mathcal{E}} \qquad \frac{\Gamma \vdash \blacksquare A_1, \ldots, \Gamma \vdash \blacksquare A_k \quad \blacksquare A_1, \ldots \blacksquare A_k \vdash B}{\Gamma \vdash \blacksquare B}\ \blacksquare_{\mathcal{I}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{I}}$$

$$\frac{\Gamma \vdash \Box A_1 \ldots \Gamma \vdash \Box A_k \ \Gamma \vdash \blacklozenge B \quad \Box A_1 \ldots \Box A_k, B \vdash \blacklozenge C}{\Gamma \vdash \blacklozenge C}\ \blacklozenge_{\mathcal{E}}$$

$$\frac{\Gamma \vdash \blacksquare A_1 \ldots \Gamma \vdash \blacksquare A_k \ \Gamma \vdash \Diamond B \quad \blacksquare A_1 \ldots \blacksquare A_k, B \vdash \Diamond C}{\Gamma \vdash \Diamond C}\ \Diamond_{\mathcal{E}}$$

Figure 8: Tense S4 rules ND first version (NDCS4)

$$\frac{\Gamma_1; \emptyset \vdash A}{\Gamma_1; \Delta \vdash \Box A}\ \Box_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \Box A \quad \Gamma_1; A; \Delta \vdash B}{\Gamma_1; \Delta \vdash B}\ \Box_{\mathcal{E}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \blacklozenge A \quad \Gamma_1; A \vdash \blacklozenge B}{\Gamma_1; \Delta \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{E}}$$

$$\frac{\Gamma; \emptyset \vdash A}{\Gamma; \emptyset \vdash \blacksquare A}\ \blacksquare_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \blacksquare A \quad \Gamma_1; A; \Delta \vdash B}{\Gamma_1; \Delta \vdash B}\ \blacksquare_{\mathcal{E}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\ \Diamond_{\mathcal{I}} \qquad \frac{\Gamma_1; \Delta \vdash \Diamond A \quad \Gamma_1; A \vdash \Diamond B}{\Gamma_1; \Delta \vdash \Diamond B}\ \Diamond_{\mathcal{E}}$$

Figure 9: biS4 rules, dual context version (ND2CS4)

$$\frac{}{\Gamma_1; \Gamma_2, A; \Delta \vdash A}\ \Box\mathrm{Id} \qquad \frac{}{\Gamma_1, A; \Gamma_2; \Delta \vdash A}\ \blacksquare\mathrm{Id}$$

$$\frac{\Gamma_1; \Gamma_2; \emptyset \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \Box A}\ \Box_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \Box A \quad \Gamma_1; \Gamma_2, A; \Delta \vdash B}{\Gamma_1; \Gamma_2; \Delta \vdash B}\ \Box_{\mathcal{E}}$$

$$\frac{\Gamma_1; \Gamma_2; \Delta \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge A}\ \blacklozenge_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge A \quad \Gamma_1; \Gamma_2; A \vdash \blacklozenge B}{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{E}}$$

$$\frac{\Gamma_1; \Gamma_2; \emptyset \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \blacksquare A}\ \blacksquare_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \blacksquare A \quad \Gamma_1, A; \Gamma_2; \Delta \vdash B}{\Gamma_1; \Gamma_2; \Delta \vdash B}\ \blacksquare_{\mathcal{E}}$$

$$\frac{\Gamma_1; \Gamma_2; \Delta \vdash A}{\Gamma_1; \Gamma_2; \Delta \vdash \Diamond A}\ \Diamond_{\mathcal{I}} \qquad \frac{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge A \quad \Gamma_1; \Gamma_2; A \vdash \blacklozenge B}{\Gamma_1; \Gamma_2; \Delta \vdash \blacklozenge B}\ \blacklozenge_{\mathcal{E}}$$

Figure 10: Dual context 2CS4 calculus (TCS4)

$$\frac{A}{\Box A}\ \Box_{Nec} \qquad \frac{A}{\blacksquare A}\ \blacksquare_{Nec} \qquad \frac{A \to B \quad A}{B}\ \mathcal{MP}$$

Figure 11: Axiomatic Rules

we are interested in, this is what we do in the next section.

3 Term Assignment

In this section we provide a term assignment to constructive tense logic with only $\Box$ and $\blacksquare$. We leave term assignments to the other varieties of tense logic with $\Diamond$ and $\blacklozenge$ for future work.

The typing rules can be found in Figure 13 with the typed equality rules in Figure 14. Here we can see that types are tense S4 formulas. The sequents have the form $\Gamma \vdash t : A$ and $\Gamma \vdash s = t : A$ where Γ is a multiset of free variables and their types denoted $x : A$, and s and t are terms with the following syntax:

$$\begin{aligned} t \quad := \quad & x \mid \lambda x : A.t \mid s\, t \mid \mathsf{let}_{\Box}\, x_1 : \Box\, A_1, \ldots, x_k : \Box\, A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t \mid \\ & \mathsf{let}_{\blacksquare}\, x_1 : \blacksquare\, A_1, \ldots, x_k : \blacksquare\, A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t \mid \mathsf{unbox}_{\Box}\ t \mid \mathsf{unbox}_{\blacksquare}\ t \end{aligned}$$

Equality is straightforward where it is apparent that the let-expressions model explicit substitutions. These substitutions are triggered when they are applied to an propositional basic intuitionistic axioms

$$
\begin{array}{ccc}
\Box(A \to B) & \to & (\Box A \to \Box B) \\
\Box(A \to B) & \to & (\blacklozenge A \to \blacklozenge B) \\
(\Box A \to A) & \wedge & (A \to \blacklozenge A) \\
(\Box A \to \Box\,\Box\, A) & \wedge & (\blacklozenge\,\blacklozenge\, A \to \blacklozenge A) \\
\blacksquare(A \to B) & \to & (\blacksquare A \to \blacksquare B) \\
\blacksquare(A \to B) & \to & (\Diamond A \to \Diamond B) \\
(\blacksquare A \to A) & \wedge & (A \to \Diamond A) \\
(\blacksquare A \to \blacksquare\blacksquare A) & \wedge & (\Diamond\Diamond A \to \Diamond A) \\
\Diamond\blacksquare A \to A & \wedge & A \to \blacksquare\Diamond A \\
\blacklozenge\,\Box\, A \to A & \wedge & A \to \Box\,\blacklozenge\, A
\end{array}
$$

Figure 12: Axioms for $\mathsf{TCS4}$

$$
\frac{}{\Gamma, x : A \vdash x : A}\mathrm{Id}
\qquad
\frac{}{\Gamma, x : \perp \vdash \mathsf{contra} : A}\perp_{\mathcal{E}}
\qquad
\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.t : A \to B}\to_{\mathcal{I}}
$$

$$
\frac{\Gamma \vdash t_1 : A \to B \quad \Gamma \vdash t_2 : A}{\Gamma \vdash t_1\, t_2 : B}\to_{\mathcal{E}}
\qquad
\frac{\Gamma \vdash t : \Box\, B}{\Gamma \vdash \mathsf{unbox}_{\Box}\ t : B}\Box_{\mathcal{E}}
$$

$$
\frac{\Gamma \vdash t_1 : \Box\, A_1, \ldots, \Gamma \vdash t_k : \Box\, A_k \quad x_1 : \Box\, A_1, \ldots, x_k : \Box\, A_k \vdash t : B}{\Gamma \vdash \mathsf{let}_{\Box}\, x_1 : \Box\, A_1, \ldots, x_k : \Box\, A_k\, \mathsf{be}\, t_1, \ldots, t_k\, \mathsf{in}\, t : \Box\, B}\Box_{\mathcal{I}}
$$

$$
\frac{\Gamma \vdash t : \blacksquare\, B}{\Gamma \vdash \mathsf{unbox}_{\blacksquare}\ t : B}\blacksquare_{\mathcal{E}}
$$

$$
\frac{\Gamma \vdash t_1 : \blacksquare\, A_1, \ldots, \Gamma \vdash t_k : \blacksquare\, A_k \quad x_1 : \blacksquare\, A_1, \ldots, x_k : \blacksquare\, A_k \vdash t : B}{\Gamma \vdash \mathsf{let}_{\blacksquare}\, x_1 : \blacksquare\, A_1, \ldots, x_k : \blacksquare\, A_k\, \mathsf{be}\, t_1, \ldots, t_k\, \mathsf{in}\, t : \blacksquare\, B}\blacksquare_{\mathcal{I}}
$$

Figure 13: TCS4 Typing Rules

$$\frac{\Gamma, x : A \vdash t_2 = s_2 : B \quad \Gamma \vdash t_1 = s_1 : A}{\Gamma \vdash (\lambda x : A.t_2)\ t_1 = [s_1/x]s_2 : B}\beta$$

$$\frac{\Gamma \vdash t_1 = s_1 : \Box A_1, \ldots, \Gamma \vdash t_k = s_k : \Box A_k \quad x_1 : \Box A_1, \ldots, x_k : \Box A_k \vdash t = s : B}{\Gamma \vdash \mathsf{unbox}_{\Box}\ (\mathsf{let}_{\Box}\ x_1 : \Box A_1, \ldots, x_k : \Box A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t) = [s_1/x_1] \ldots [s_k/x_k]s : B}\Box$$

$$\frac{\Gamma \vdash t_1 = s_1 : \blacksquare A_1, \ldots, \Gamma \vdash t_k = s_k : \blacksquare A_k \quad x_1 : \blacksquare A_1, \ldots, x_k : \blacksquare A_k \vdash t = s : B}{\Gamma \vdash \mathsf{unbox}_{\blacksquare}\ (\mathsf{let}_{\blacksquare}\ x_1 : \blacksquare A_1, \ldots, x_k : \blacksquare A_k\ \mathsf{be}\ t_1, \ldots, t_k\ \mathsf{in}\ t) = [s_1/x_1] \ldots [s_k/x_k]s : B}\blacksquare$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t = t : A}\mathrm{refl} \qquad \frac{\Gamma \vdash t_2 = t_1 : A}{\Gamma \vdash t_1 = t_2 : A}\mathrm{sym} \qquad \frac{\Gamma \vdash t_1 = t_2 : A \quad \Gamma \vdash t_2 = t_3 : A}{\Gamma \vdash t_1 = t_3 : A}\mathrm{trans}$$

Figure 14: TCS4 Equality Rules

unbox-expression.

We have the following basic properties of this term assignment.

Lemma 1 (Substitution for Typing)**.** *If* $\Gamma_1 \vdash t_1 : A$*, and* $\Gamma_1, x : A, \Gamma_2 \vdash t_2 : B$*, then* $\Gamma_1, \Gamma_2 \vdash [t_1/x]t_2 : B$*.*

Proof. This proof holds by straightforward induction on the form of the assumed typing derivation. Please see Appendix A.1.1 for the proof. □

Lemma 2 (Weakening)**.** *If* $\Gamma_1, \Gamma_2 \vdash t : B$*, then* $\Gamma_1, x : A, \Gamma_2 \vdash t : B$*.*

Proof. This proof holds by straightforward induction on the form of the assumed typing derivation. Please see Appendix A.1.2 for the proof. □

4 The Categorical Model

There is not much essentially new in what we discuss here about the tense logic based on CS4. Similar ideas were discussed by Ghilardi and Meloni [13], Makkai and Reyes [18] and more recently in by Dzik et al [7, 9] and Menni and Smith [19].

The upshot of our discussion is that the categorical model we advance is a cartesian closed category endowed with two adjunctions, corresponding to the (limited) universal and existential quantifications relative to the past and to the future that correspond to the two sets of necessity and possibility operators.

This setting is though different enough from the precursors we know about, to justify this note. First, as discussed elsewhere [5], we see no reason for the monads/comonads emerging from this setting to be **idempotent** operators, as they are in [13] or [18] (the idempotency simplification does not seen warranted by the proof theory). Secondly we see no reason to take our models as part of toposes, as we are not interested in the extra structure provided by toposes. However, we also see no reason to confine ourselves to algebraic models such as Heyting algebras with operators, as degenerate posetal categories, as both [7] and [19] do. Different proofs of the same theorem are important to us as they correspond to different morphisms in the category between the same objects. Thus we are interested in **proof relevant** semantics, not simply provability.

We build our main definition in stages. To begin with, a categorical model of propositional intuitionistic logic is a **cartesian closed** category $\mathcal{C}$ with coproducts. Then we recall from [5] that to model a pair of modalities using dual contexts we need a *monoidal* adjunction.

Definition 3 (adjoint model)**.** *An adjoint categorical model of dual context modal logic* DCS4 *consists of the following data:*

1. *A cartesian closed category with coproducts* $(\mathcal{C}, 1, 0, \times, +, \to)$*;*
2. *A monoidal adjunction* $F \dashv G$*, where* (F, m) *and* $(G, n)\colon \mathcal{C} \longrightarrow \mathcal{C}$ *are monoidal functors such that their composition* GF *is a monoidal comonad, written as* $\Box$*;*
3. *The monad* $(\Diamond, \eta, \mu, \mathsf{st}_{A,B})$*, induced by the adjunction* $F \dashv G$*, is* $\Box$*-strong.*

Recall that a *monoidal* comonad $\Box$ implies that there is a natural transformation $m\colon \Box A \times \Box B \to \Box(A \times B)$ (and $m_\top\colon \top \to \Box\top$) satisfying the coherence conditions described in page 23 of [5]. Recall as well that by a monad being $\Box$-strong, we mean that there is a *strength* natural transformation $st_{A,B}\colon \Box A \times \Diamond B \to \Diamond(\Box A \times B)$ satisfying the four equations in page 27 of [5]. These two natural transformations are required to model the Fisher-Servi axioms, which are a weakening of the duality between $\Box$ and $\Diamond$ that the classical modalities satisfy.

Finally we consider two pairs of modalities (or two adjunctions), intertwined, as in tense logic.

Definition 4 (tense calculus model)**.** *A categorical model of tense calculus dual context modal logic* TCDS4 *is a cartesian closed category* $\mathcal{C}$ *as above, together with two intertwined adjunctions* $(\blacklozenge \dashv \Box, \Diamond \dashv \blacksquare)$*. The adjunctions* $(\blacklozenge \dashv \Box)$ *and* $(\Diamond \dashv \blacksquare)$ *on* $\mathcal{C}$ *are connected by the Fisher-Servi axioms, namely* $\Diamond(A \to B) \to (\blacksquare A \to \Diamond B)$ *and* $(\Diamond A \to \blacksquare B) \to \blacksquare(A \to B)$*, as well as* $\blacklozenge(A \to B) \to (\Box A \to \blacklozenge B)$ *and* $(\blacklozenge A \to \Box B) \to \Box(A \to B)$*.*

This model is more general than the system, $\mathsf{TCS4}$, given above in that it contains two possibility modalities, which we do not deal with, in the type theory. These possibility operators could be treated as syntactic sugar for $\neg\Box\neg A$ (and respectively $\neg\blacksquare\neg A$), as they usually are in Intuitionistic Linear Logic, for instance. We refrain from doing so explicitly and prefer to consider the necessity-only fragment in the type theory, as this allows us to bypass the discussion of which possibility modality is more appropriate for each setting. More importantly it allows us to dodge the question of how to provide a Curry-Howard categorical interpretation for what we called Simpson-style modal S4. Thus the model should be seen as an over approximation. We give the more general model here to set the stage for future work.

Categorical soundness is proved, as usual, checking the natural deduction rules preserve validity of the constructions used, i.e function spaces, products, coproducts and the two adjunctions.

Define an interpretation $[\![\_]\!] : \mathsf{TCS4} \to \mathcal{C}$ which takes the types and sequents of $\mathsf{TCS4}$ (over a basic set of types) to a model $\mathcal{C}$ as follows:

$$
\begin{aligned}
[\![p]\!] &= I(p) \text{ for } p \text{ a base type} \\
[\![\top]\!] &= \top \\
[\![A \to B]\!] &= [\![A]\!] \to [\![B]\!] \\
[\![\Box A]\!] &= FG([\![A]\!]) \\
[\![\blacksquare A]\!] &= F'G'([\![A]\!])
\end{aligned}
$$

We extend this interpretation to lists of types by saying that for a list $A_1, ..., A_n$ of types, the interpretation is the product of the interpretations $[\![A_1, ...A_n]\!] = [\![A_1]\!] \times \ldots \times [\![A_n]\!]$. The interpretation will take a sequent $\Gamma \vdash t : A$ to an arrow $[\![\Gamma \vdash t : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$ in the tense modal category.

Theorem 5. *The type theory* $\mathsf{TCS4}$ *has sound models provided by the structures* $\mathcal{C}$ *defined above. In other words, given a tense adjoint modal category* $\mathcal{C}$*, using the above interpretation, the following hold:*

- *Assume* $\Gamma \vdash t : A$ *in* $\mathsf{TCS4}$*. Then* $[\![\Gamma \vdash t \colon A]\!]$ *is a morphism with domain* $[\![\Gamma]\!]$ *and codomain* $[\![A]\!]$*;*
- *Assume* $\Gamma \vdash t = s \colon A$*. Then* $[\![\Gamma \vdash t \colon A]\!] = [\![\Gamma \vdash s \colon A]\!]$*.*

Proof. The first part holds by induction on $\Gamma \vdash t : A$, and the second by induction on $\Gamma \vdash t = s : A$, but uses the first part. Please see Appendix A.1.3 for the proof. $\square$

We have completeness of the tense modal categories when the model is restricted to box modalities only.

Theorem 6. *The adjoint modal models are complete in the appropriate sense for the type theory* TCS4*. This is to say, if we have equality of the interpretations* $[\![\Gamma \vdash t\colon A]\!] = [\![\Gamma \vdash s\colon A]\!]$ *(where* $[\![\ ]\!]$ *is the interpretation defined above) in the tense modal category* $\mathcal{C}$ *for any derived sequents* $\Gamma \vdash t\colon A$ *and* $\Gamma \vdash s\colon A$ *then we can derive the equation in the type theory* TCS4 $\quad \Gamma \vdash t = s\colon A$*.*

Proof. This result can be shown by constructing a cartesian closed category with coproducts and two comonads, one for $\Box$ and one for $\blacksquare$, internal to TCS4 where the objects are types and the morphisms are α-equivalence classes of terms in context $\Gamma \vdash t : A$. This category is called the syntactic category. Please see Appendix A.1.4 for the remainder of the proof. $\Box$

Categorical completeness requires providing an equivalence relation in the Lindenbaum algebra of the formulae, as usual in algebraic semantics. The basic calculations, for traditional algebraic semantics in Heyting algebras were provided, for instance, by Figallo et al in [11] or Dzik et al in [8]. *Mutatis mutantis* these calculations will apply for our version of the system (no distribution of diamonds over disjunctions, no definibility of diamonds in terms of negated boxes).

5 Conclusions

We have described a tense version of constructive temporal logic, conceived as a basic category of propositions, together with two adjunctions, corresponding to two kinds of necessity modalities, in the future and in the past. This system is based on traditional work of Ewald in [10], where we simply do the modifications required to account for the categorical model desired. This work is somewhat inspired by recent work on Functional Reactive Programming (FRP) by Jeltsch [16] and Jeffrey [15], independently. Both of these works consider Curry-Howard correspondents to temporal logic, but they tend to concentrate on the *next* temporal operator, originally considered in LTL (Linear Temporal Logic), as suggested by Davies [6]. The temporal operators we consider are more abstract and one can hope that they may shed some light on the issues of FRP. But this is future work.

References

[1] Natasha Alechina, Michael Mendler, Valeria De Paiva, and Eike Ritter. Categorical and Kripke semantics for constructive S4 modal logic. In *Computer Science Logic*, pages 292–307. Springer, 2001.

[2] Ryuta Arisaka, Anupam Das, and Lutz Straßburger. On Nested Sequents for Constructive Modal Logics. Logic Journal of the IGPL, 22, pp. 992–1018, 2014.

[3] Andrew Barber. *Linear Type Theories, Semantics and Action Calculi.* LFCS, University of Edinburgh, 1997.

[4] Nick Benton. A mixed linear and non-linear logic: Proofs, terms and models. In *Computer Science Logic*, pages 121–135. Springer, 1995.

[5] G.M. Bierman and V.C.V. de Paiva. On an Intuitionistic Modal Logic. *Studia Logica*, 65(3):383–416, 2000.

[6] Rowan Davies. A temporal-logic approach to binding-time analysis. In *Logic in Computer Science, 1996. LICS'96. Proceedings., Eleventh Annual IEEE Symposium on*, pages 184–195. IEEE, 1996.

[7] W. Dzik, J. Järvinen, and M. Kondo. Intuitionistic logic with two Galois connections combined with Fischer Servi axioms. *ArXiv e-prints*, August 2012.

[8] Wojciech Dzik, Jouni Järvinen, and Michiro Kondo. Intuitionistic propositional logic with galois connections. *Logic Journal of IGPL*, 18(6):837–858, 2010.

[9] Wojciech Dzik, Jouni Järvinen, and Michiro Kondo. Characterizing intermediate tense logics in terms of galois connections. *Logic Journal of IGPL*, 22(6):992–1018, 2014.

[10] W. B. Ewald. Intuitionistic tense and modal logic. *The Journal of Symbolic Logic*, 51:166–179, 3, 1986.

[11] Aldo V. Figallo and Gustavo Pelaitay. An algebraic axiomatization of the Ewald's intuitionistic tense logic. *Soft Computing*, 18(10):1873–1883, 2014.

[12] Neil Ghani, Valeria de Paiva, and Eike Ritter. Explicit Substitutions for Constructive Necessity. In *ICALP International Conference on Automata, Languages and Programming*, 1998.

[13] Silvio Ghilardi and GC Meloni. Modal and tense predicate logic: Models in presheaves and categorical conceptualization. In *Categorical algebra and its applications*, pages 130–142. Springer, 1988.

[14] Jean Goubault-Larrecq and Eric Goubault. Order-Theoretic, Geometric and Combinatorial Models of Intuitionistic S4 Proofs. In *In Intuitionistic Modal Logic and Applications (IMLA'99)*, 1999.

[15] Alan Jeffrey. LTL types FRP: Linear-time Temporal Logic Propositions as Types, Proofs as Functional Reactive Programs. In *ACM Workshop Programming Languages meets Program Verification*. ACM, 2012.

[16] Wolfgang Jeltsch. Towards a Common Categorical Semantics for Linear-Time Temporal Logic and Functional Reactive Programming. *Electronic Notes in Theoretical Computer Science, Proceedings of the 28th Conference on the Mathematical Foundations of Programming Semantics (MFPS XXVIII)*, 286:229 – 242, 2012.

[17] Ph. Kremer and G. Mints. Dynamic topological logic. *Annals of Pure and Applied Logic*, 131(1-3):133–158, 2005.

[18] M. Makkai and G.E. Reyes. Completeness results for intuitionistic and modal logic in a categorical setting. *Annals of Pure and Applied Logic*, 72:25–101, 1995.

[19] M. Menni and C. Smith. Modes of Adjointness. *Journal of Philosophical Logic*, 43(2-3):365–391, 2014.

[20] Grigori Mints. *A short introduction to intuitionistic logic*. Kluwer Academic Publishers, Norwell, MA, USA, 2000.

[21] Masao Ohnishi and Kazuo Matsumoto. Gentzen method in modal calculi. *Osaka Math. J.*, 9(2):113–130, 1957.

[22] Jeniffer Paykin and Steve Zdancewic. A linear/producer/consumer model of classical linear logic (extended abstract). In *Third International Workshop on Linearity, LINEARITY*, 2012.

[23] Dag Prawitz. *Natural Deduction, volume 3 of Stockholm Studies in Philosophy*. Almqvist and Wiksell, 1965.

[24] A. N. Prior. *Time and Modality*. Oxford University Press, 1957. Based on his 1956 John Locke lectures.

[25] Alex K Simpson. *The proof theory and semantics of intuitionistic modal logic*. University of Edinburgh. College of Science and Engineering, School of Informatics, 1994.

[26] Lutz Straßburger. Cut elimination in nested sequents for intuitionistic modal logics. In *FOSSACS*, 2013.

A Appendix

A.1 Proofs

A.1.1 Proof of Substitution for Typing

Lemma (Substitution for Typing)**.** *If* $\Gamma_1 \vdash t_1 : A$*, and* $\Gamma_1, x : A, \Gamma_2 \vdash t_2 : B$*, then* $\Gamma_1, \Gamma_2 \vdash [t_1/x]t_2 : B$*.*

Proof. Suppose $\Gamma_1 \vdash t_1 : A$ and $\Gamma_1, x : A, \Gamma_2 \vdash t_2 : B$. We case split on the structure of the latter, but only show the non-trivial cases. All other cases are similar.

Case Identity.

$$\frac{}{\Gamma_1, x : A, \Gamma_2 \vdash y : C}\ \textsc{Id}$$

In this case $t_2 = y$ and $B = C$. We are not sure if $x = y$, thus, we must consider the case when they are and are not equal.

Suppose $x \neq y$. Then $[t_1/x]t_2 = [t_1/x]y = y$ by the definition of substitution. In addition, it must be the case that either $y : C \in \Gamma_1$ or $y : C \in \Gamma_2$. This implies that $\Gamma_1, \Gamma_2 \vdash y : C$ or $\Gamma_1, \Gamma_2 \vdash [t_1/x]t_2 : B$ hold.

Now suppose $x = y$. Then $A = B$, and $[t_1/x]t_2 = [t_1/x]x = t_1$ by the definition of substitution. Thus, $\Gamma_1, \Gamma_2 \vdash [t_1/x]t_2 : B$ holds, because we know $\Gamma_1, \Gamma_2 \vdash t_1 : A$.

Case Implication Introduction.

$$\frac{\Gamma_1, x : A, \Gamma_2, y : C_1 \vdash t : C_2}{\Gamma_1, x : A, \Gamma_2 \vdash \lambda y : C_1.t : C_1 \to C_2}\ \to_{\mathcal{I}}$$

In this case $B = C_1 \to C_2$ and $t_2 = \lambda y : C.t$. By the induction hypothesis we know $\Gamma_1, \Gamma_2, y : C_1 \vdash [t_1/x]t : C_2$, and then by reapplying the rule we know $\Gamma_1, \Gamma_2 \vdash \lambda y : C_1.[t_1/x]t : C_2$ holds. However, by the definition of substitution we know $\lambda y : C_1.[t_1/x]t = [t_1/x](\lambda y : C_1.t)$, and thus, we obtain our result.

Case Implication Elimination.

$$\frac{\Gamma_1, x : A, \Gamma_2 \vdash t_1' : C_1 \to C_2 \quad \Gamma_1, x : A, \Gamma_2 \vdash t_2' : C_1}{\Gamma_1, x : A, \Gamma_2 \vdash t_1'\, t_2' : C_2} \to_{\mathcal{E}}$$

We now have that $B = C_2$ and $t_2 = t_1'\, t_2'$. By the induction hypothesis we know that $\Gamma_1, \Gamma_2 \vdash [t_1/x]t_1' : C_1 \to C_2$ and $\Gamma_1, \Gamma_2 \vdash [t_1/x]t_2' : C_1$ both hold. Then by reapplying the rule we obtain that $\Gamma_1, \Gamma_2 \vdash ([t_1/x]t_1')\,([t_1/x]t_2') : C_2$, and thus, by the definition of substitution $\Gamma_1, \Gamma_2 \vdash [t_1/x](t_1'\, t_2') : C_2$ holds.

Case $\Box$ Introduction.

$$\frac{\Gamma_1, x : A, \Gamma_2 \vdash t_1' : \Box\, C_1,\, \dots, \Gamma_1, x : A, \Gamma_2 \vdash t_k' : \Box\, C_k \quad x_1 : \Box\, C_1,\, \dots, x_k : \Box\, C_k \vdash t : C}{\Gamma_1, x : A, \Gamma_2 \vdash \mathsf{let}_\Box\, x_1 : \Box\, C_1,\, \dots, x_k : \Box\, C_k\, \mathsf{be}\, t_1',\, \dots, t_k'\, \mathsf{in}\, t : \Box\, C} \Box_{\mathcal{I}}$$

In this case $B = \Box\, C$ and $t_2 = \mathsf{let}_\Box\, x_1 : \Box\, C_1,\, \dots, x_k : \Box\, C_k\, \mathsf{be}\, t_1',\, \dots, t_k'\, \mathsf{in}\, t$. By the induction hypothesis we know that

$$\Gamma_1, \Gamma_2 \vdash [t_1/x]t_1' : \Box\, C_1, \ldots, \Gamma_1, x : A, \Gamma_2 \vdash [t_1/x]t_k' : \Box\, C_k$$

all hold. Then by reapplying the rule we know that

$$\Gamma_1, \Gamma_2 \vdash \mathsf{let}_\Box\, x_1 : \Box\, C_1,\, \dots, x_k : \Box\, C_k\, \mathsf{be}\, [t_1/x]t_1',\, \dots, [t_1/x]t_k'\, \mathsf{in}\, t : \Box\, C,$$

but by the definition of substitution and the fact that $[t_1/x]t = t$ because t does not depend on x we know that

$$\Gamma_1, \Gamma_2 \vdash [t_1/x](\mathsf{let}_\Box\, x_1 : \Box\, C_1,\, \dots, x_k : \Box\, C_k\, \mathsf{be}\, t_1',\, \dots, t_k'\, \mathsf{in}\, t) : \Box\, C$$

holds.

$\Box$

A.1.2 Proof of Weakening

Lemma (Weakening)**.** *If* $\Gamma_1, \Gamma_2 \vdash t : B$, *then* $\Gamma_1, x : A, \Gamma_2 \vdash t : B$.

Proof. This proof is by induction on the form of $\Gamma_1, \Gamma_2 \vdash t : B$. We only show a few cases, because the others are similar.

Case Identity.

$$\frac{}{\Gamma_1, \Gamma_2, y : C \vdash y : C} \textsc{Id}$$

In this case we have that $B = C$ and $t = y$. We must show that $\Gamma_1, x : A, \Gamma_2, y : C \vdash y : C$ holds, but this clearly holds by reapplying the rule.

Case $\Box$ Introduction.

$$\frac{\Gamma_1, \Gamma_2 \vdash t_1 : \Box\, C_1, \ldots, \Gamma_1, \Gamma_2 \vdash t_k : \Box\, C_k \quad x_1 : \Box\, C_1, \ldots, x_k : \Box\, C_k \vdash t' : C}{\Gamma_1, \Gamma_2 \vdash \mathsf{let}_\Box\, x_1 : \Box\, C_1, \ldots, x_k : \Box\, C_k\, \mathsf{be}\, t_1, \ldots, t_k\, \mathsf{in}\, t' : \Box\, C}\,\Box_{\mathcal{I}}$$

This case is similar to the previous case. First, apply the induction hypothesis to the left-most premise, and then reapply the rule.

$\Box$

A.1.3 Proof of Soundness of TCS4

Theorem. *The type theory* TCS4 *has sound models provided by the structures $\mathcal{C}$ defined above. In other words, given a tense adjoint modal category $\mathcal{C}$, using the above interpretation, the following hold:*

- *Assume $\Gamma \vdash t : A$ in* TCS4*. Then $[\![\Gamma \vdash t\colon A]\!]$ is a morphism with domain $[\![\Gamma]\!]$ and codomain $[\![A]\!]$;*
- *Assume $\Gamma \vdash t = s\colon A$. Then $[\![\Gamma \vdash t\colon A]\!] = [\![\Gamma \vdash s\colon A]\!]$.*

Proof. The first part holds by induction on $\Gamma \vdash t : A$, and the second by induction on $\Gamma \vdash t = s : A$. We give a few cases of each part, as the others are similar. Throughout the proof we drop semantic brackets on objects, and we assume, without loss of generality, that the interpretation of contexts are left associated. We begin with the first part.

Case Identity.

$$\frac{}{\Gamma, x : A \vdash x : A}\,\mathrm{Id}$$

We need to provide a morphism $\Gamma \times A \xrightarrow{f} A$ and we choose $f = \pi_2$ (the 2nd projection), as usual.

Case Implication Introduction.

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.t : A \to B}\,\to_{\mathcal{I}}$$

By the induction hypothesis we know that there is a morphism $\Gamma \times A \xrightarrow{f} B$. Then we need to find a morphism $\Gamma \xrightarrow{g} (A \to B)$. Choose $g = \mathsf{curry}(f)$ where $\mathsf{curry} : \mathsf{Hom}_{\mathcal{C}}(A \times B, C) \longrightarrow \mathsf{Hom}_{\mathcal{C}}(A, B \to C)$ is a natural isomorphism that exists because $\mathcal{C}$ is closed.

Case $\Box$ Introduction.

$$\frac{\Gamma \vdash t_1 : \Box A_1, \ldots, \Gamma \vdash t_k : \Box A_k \quad x_1 : \Box A_1, \ldots, x_k : \Box A_k \vdash t : B}{\Gamma \vdash \mathsf{let}_\Box\, x_1 : \Box A_1, \ldots, x_k : \Box A_k \,\mathsf{be}\, t_1, \ldots, t_k \,\mathsf{in}\, t : \Box B}\Box_{\mathcal{I}}$$

By the induction hypothesis we have the family of morphisms $\Gamma \xrightarrow{f_1} \Box A_1, \ldots, \Gamma \xrightarrow{f_k} \Box A_k$, and a given morphism $\Box A_1 \times \cdots \times \Box A_k \xrightarrow{f} B$. We need to find a morphism $\Gamma \xrightarrow{g} B$. As in previous work, we choose $g = \langle f_1; \delta_{A_1}, \ldots, f_k; \delta_{A_k} \rangle; \mathsf{m}; \Box f$, where $\langle -, - \rangle : \mathsf{Hom}_{\mathcal{C}}(\Gamma, \Box A_1) \times \cdots \times \mathsf{Hom}_{\mathcal{C}}(\Gamma, \Box A_k) \longrightarrow \mathsf{Hom}_{\mathcal{C}}(\Gamma, \Box A_1 \times \cdots \times \Box A_k)$ exists because $\mathcal{C}$ is cartesian and we make the simplifying assumption that $\Box$ is an endofunctor.

Case $\Box$ Elimination.

$$\frac{\Gamma \vdash t : \Box B}{\Gamma \vdash \mathsf{unbox}_\Box\ t : B}\Box_{\mathcal{E}}$$

By the induction hypothesis there is a morphism $\Gamma \xrightarrow{f} \Box B$. It suffices to find a morphism $\Gamma \xrightarrow{g} B$. Choose $g = f; \eta_B$ where $\eta_B : \Box B \longrightarrow B$ is the unit of the adjunction.

We now turn to the second part:

Case Unboxing $\Box$.

$$\frac{\Gamma \vdash t_1 = s_1 : \Box A_1, \ldots, \Gamma \vdash t_k = s_k : \Box A_k \quad x_1 : \Box A_1, \ldots, x_k : \Box A_k \vdash t = s : B}{\Gamma \vdash \mathsf{unbox}_\Box\ (\mathsf{let}_\Box\, x_1 : \Box A_1, \ldots, x_k : \Box A_k \,\mathsf{be}\, t_1, \ldots, t_k \,\mathsf{in}\, t) = [s_1/x_1] \ldots [s_k/x_k]s : B}\Box$$

Using the interpretations given above we must show that:

$$\langle f_1; \delta_{A_1}, \ldots, f_k; \delta_{A_k} \rangle; \mathsf{m}; \Box f; \eta_B = \langle f_1, \ldots, f_k \rangle; f : \Gamma \longrightarrow B.$$

This holds by the following equational reasoning:

$$\begin{array}{lll}
\langle f_1; \delta_{A_1}, \ldots, f_k; \delta_{A_k} \rangle; \mathsf{m}; \Box f; \eta_B & = & \langle f_1; \delta_{A_1}, \ldots, f_k; \delta_{A_k} \rangle; \mathsf{m}; \eta; f \\
& = & \langle f_1; \delta_{A_1}, \ldots, f_k; \delta_{A_k} \rangle; (\eta_{A_1} \times \cdots \times \eta_{A_k}); f \\
& = & \langle f_1; \delta_{A_1}; \eta_{A_1}, \ldots, f_k; \delta_{A_k}; \eta_{A_k} \rangle; f \\
& = & \langle f_1, \ldots, f_k \rangle; f
\end{array}$$

$\Box$

A.1.4 Proof of Completeness for $\mathsf{TCS4}$

Theorem. *The adjoint modal models are complete in the appropriate sense for the type theory* $\mathsf{TCS4}$*. This is to say, if we have equality of the interpretations* $[\![\Gamma \vdash t\colon A]\!] = [\![\Gamma \vdash s\colon A]\!]$ *(where* $[\![\ ]\!]$ *is the interpretation defined above) in the tense modal category* $\mathcal{C}$ *for any derived sequents* $\Gamma \vdash t\colon A$ *and* $\Gamma \vdash s\colon A$ *then we can derive the equation in the type theory* $\mathsf{TCS4}$ $\quad \Gamma \vdash t = s\colon A$.

Proof. This result can be shown by constructing a cartesian closed category with two monoidal comonads, one for $\Box$ and one for $\blacksquare$, internal to the type theory $\mathsf{TCS4}$ where the objects are types and the morphisms are α-equivalence classes of terms in context $\Gamma \vdash t : A$. This category is called the syntactic category for the $\mathsf{TCS4}$ type theory.

Showing that this syntactic category is cartesian closed is well known, but we illustrate the proof by describing the case of the $\Box$ comonad.

We denote a morphism by the α-equivalence class:

$$[\vec{x}, t]^{\vec{A},B} = [\vec{x} : \vec{A} \vdash t : B]$$

We then have the following definitions:

- (Identity) $\mathsf{id} = [x, x]^{A,A}$
- (Composition) Given morphisms $[\vec{x}, t]^{\vec{A},B_i}$ and $[\vec{y}, t']^{\vec{B},C}$, their composition $[\vec{x}, t]^{\vec{A},B_i}; [\vec{y}, t']^{\vec{B},C} = [\vec{x_{i-1}}, \vec{y}, \vec{x_{i+1}}, [t/x_i]t']^{\vec{A},C}$.
- (Equality) Two parallel morphisms $[\vec{x}, t]^{\vec{A},B}$ and $[\vec{x}, t']^{\vec{A},B}$ are equal if and only if $\vec{x} : \vec{A} \vdash t = t' : B$.

Using basic facts about substitution one can show that composition preserves identity and is associative.

We first must show that $\Box$ is an endofunctor on the syntactic category. Suppose we have the morphism $[\vec{x}, t]^{\vec{A},B}$. Then we must construct a morphism $[\vec{y}, t']^{\overrightarrow{\Box A},\Box B}$. The latter morphism can be defined in two steps. The first is to change the $\vec{A}$ to $\overrightarrow{\Box A}$:

$$[\vec{y}, \wedge \mathsf{unbox}_\Box\ y_i]^{\overrightarrow{\Box A},\vec{A}}; [\vec{x}, t]^{\vec{A},\vec{B}} = [\vec{y}, [\mathsf{unbox}_\Box\ y_i/x_i]t]^{\overrightarrow{\Box A},\vec{B}}$$

The second step is to change B into $\Box B$:

$$[\vec{y}, \mathsf{let}_\Box\ \vec{y} : \overrightarrow{\Box A}\ \mathsf{be}\ \vec{y}\ \mathsf{in}\ [\mathsf{unbox}_\Box\ y_i/x_i]t)]^{\overrightarrow{\Box A},\Box B}$$

Straightforward calculations show that this construction preserves identities and composition.

The unit of the comonad is defined as $[y, \mathsf{unbox}_{\Box}\ y]^{\Box A, A}$. Next we need to define a morphism between $\Box A$ and $\Box \Box A$:

$$[y, \mathsf{let}_{\Box}\ y : \Box A\ \mathsf{be}\ y\ \mathsf{in}\ y]^{\Box A, \Box\Box A}$$

Finally, using these constructions it is possible to show the usual diagrams defining the comonad $\Box$ hold. The definition for $\blacksquare$ is similar. $\Box$

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