A proof theoretical view of ecumenical systems

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Abstract

Much has been said about intuitionistic and classical logical systems since Gentzen's seminal work. Recently Prawitz, and others, have been discussing the putting together Gentzen's systems for classical and intuitionistic (propositional) logic in a single system, that Prawitz calls the Ecumenical System. In this work we will present an ecumenical sequent calculus and state some proof theoretical properties of the system. This complete different approach for a unified system enabling both classical and intuitionistic features should shed some light not only on the logics themselves, but also in their semantical interpretation as well as proof theoretical properties that can arise from combining logical systems.

Keywords: Intuitionistic logic, classical logic, ecumenical systems, proof theory.

1 Introduction

In 1935 Gerhard Gentzen introduced sequent calculi for classical (LK) and intuitionistic (LJ) first order logic [Gen69] saying that this was "in order to be able to enunciate and prove the Haupsatz in a convenient form". Gentzen thought he had "to provide a logical calculus specially suited for this purpose", because his favourite system, natural deduction (given by systems NK and NJ for classical and intuitionistic logics), could not be used to produce a proof.

According to Gentzen, for the purpose of proving the Haupsatz "the natural calculus proved unsuitable". As he explains

> "for although it [the natural deduction calculus] already contains the properties essential to the validity of the Haptsatz, it does so only with respect to its intuitionistic form, in view of the fact that the law of excluded middle occupies a special position in relation to these properties".

In the case of LK, Gentzen continues

> "there exists complete symmetry between $\wedge$ and $\vee$, $\forall$ and $\exists$. All of the connectives $\wedge$, $\vee$, $\forall$, $\exists$ and $\neg$ have, to a large extend, equal status in the system: no connective ranks notably above any other connective. The special position of the negation, in particular, which constituted a troublesome exception in the natural calculus [natural deduction], has been completely removed in a seemingly magical way. The manner in which this observation is expressed is undoubtedly justified since I myself was completely surprised by this property of the 'LK-calculus' when first formulating that

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calculus. The 'law of the excluded middle' and the 'elimination of double negation' are implicit in the new inference schemata - the reader may convince himself of this by deriving both of them within the new calculus - but they have become completely harmless and no longer play the least special role in the consistency proof that follows."

Thirty years later, Prawitz showed, in his doctoral work [Pra65], how to actually prove the Haupsatz for natural deduction. Gentzen and Prawitz are perhaps the main founders of Proof Theory, the branch of Logic, which would get a large boost from Computer Science. Although NJ and NK together with LJ and LK have been staples of proof theory ever since, every new wave of different proof systems (Dependent Type Theory, Linear Logics, Display Calculi, Deep Inference, Deduction Modulo, etc) has improved our understanding of the basic systems.

In particular, when considering sequent based systems for intuitionistic logic, along with Gentzen's LJ, in the 50's Maehara introduced [Mae54] the system mLJ, which is a single conclusion system only in the rules for right implication and right universal quantification. Another thirty years later, using ideas from Girard's linear logic and resource sensitivity, Pereira and de Paiva introduced FIL (Full Intuitionistic Logic) [dPP05], provably equivalent to LJ, where an indexing device allows us to keep track of dependency relations between formulae. Extensions of sequent systems considering semantic labels where considered in [Vig00], while in [Fit14] Fitting related such labels with nestings and indexed tableaux. In [LPR18] it was shown how to relate nestings with sequent systems in a general way, so intuitionistic logic is now understood from various different proof theoretical and semantical points of view.

One especially important perspective is that of the Curry-Howard correspondence [How80], which gradually became the most natural embodiment of the ideas underlying intuitionistic logic. This correspondence underlies most of the work relating type systems to programming languages design and as such it is one of the reasons why proof theory has found natural growing grounds in theoretical Computer Science. The basic idea here is that intuitionistic types correspond to propositions, while lambda-terms (or programs) correspond to proofs/derivations, in such a way that evaluation of the term/proof, corresponds to normalisation of the proof. In the classical case, $\lambda\mu$-calculus [Par92] introduces two extra operators, corresponding to continuations.

Proof theoretically, the difference between intuitionistic and a classical sequent systems is given by some kind of restriction on contexts. Hence it is reasonable to ask if it is possible to naturally combine classical and intuitionistic systems, so that they can live peacefully in a single system. Citing Girard [Gir87]

"By the turn of this century the situation concerning logic was quite simple: there was basically one logic (classical logic) which could be used (by changing the set of proper axioms) in various situations. Logic was about pure reasoning. Brouwers criticism destroyed this dream of unity: classical logic was not suited for constructive features and therefore it lost its universality. Now by the end of the century we are faced with an incredible number of logics-some of them only named logic by antiphrasis, some of them introduced on serious grounds. Is logic still about pure reasoning? In other words, could there by a way to reunify logical systems- let us say those systems with a good sequent calculus -into a single sequent calculus? Is it possible to handle the (legitimate) distinction classical/intuitionistic not through a change of system, but through a change of formulas? Is it possible to obtain classical effects by a restriction to classical formulas? Etc."

Since then, several approaches have been proposed for combining intuitionistic and classical logics (see *e.g.* [LM11, Dow16]), most of them inspired by Girard's polarised system LU [Gir93]. More recently, Prawitz chose a completely different approach by proposing a natural deduction *Ecumenical System* [Pra16]. While it also took into account meaning-theoretical considerations, it is more focused on investigating the philosophical significance of the fact that classical logic can be translated into intuitionistic logic.

In this work we will present two sequent calculi for the ecumenical system: LEci, given by a direct transformation from the natural deduction system, and its less bureaucratic version, mLEci′. While the first makes heavy use of negations, the second is intended to be *purer*, in the sense that it introduces, bottom up, less negations. Since the ecumenical approach is recent and systems combining classical and intuitionistc logics are often given in the sequent calculus presentation, the first step of moving from natural deduction to the sequent calculus formulation seems to be mandatory for comparing systems.

It is worth noticing that this is an ongoing work, meaning that this is a starting point for very promising results, still in development. But it is important to notice that we already have some positive and negative results, that will most probably open doors for a better understanding not only of the logics themselves, but also of the *relationship* between their proof systems.

2 Ecumenical natural deduction system

In 2015 Dag Prawitz proposed an Ecumenical system [Pra16], where classical and intuitionistic logic could coexist in peace. Classical and intuitionistic would share the universal quantifier, conjunction, negation and the constant for the absurd, but they would each have their own existential quantifier, disjunction and implication, with different meanings. Prawitz main idea is that these different meanings are given by a semantical framework that can be accepted by both parties. In this work, we will deal only with the *propositional* part of the system, that is, we will not concern quantifiers.

The language $\mathcal{L}$ used for ecumenical systems is described as follows. Classical and intuitionistic constants $(p, q, \ldots)$

$$
\frac{\Gamma \Rightarrow A \to_i B \quad \Gamma \Rightarrow A}{\Gamma \Rightarrow B}\ \to_i E
\qquad
\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to_i B}\ \to_i I
\qquad
\frac{\Gamma \Rightarrow A \to_c B \quad \Gamma \Rightarrow A \quad \Gamma \Rightarrow \neg B}{\Gamma \Rightarrow \bot}\ \to_c E
$$

$$
\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \to_c B}\ \to_c I
\qquad
\frac{\Gamma \Rightarrow A_1 \wedge A_2}{\Gamma \Rightarrow A_j}\ \wedge E_j
\qquad
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge I
$$

$$
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow \neg A}{\Gamma \Rightarrow \bot}\ \neg E
\qquad
\frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A}\ \neg I
\qquad
\frac{\Gamma \Rightarrow A \vee_i B \quad \Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma \Rightarrow C}\ \vee_i E
$$

$$
\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee_i A_2}\ \vee_i I
\qquad
\frac{\Gamma \Rightarrow A \vee_c B \quad \Gamma \Rightarrow \neg A \quad \Gamma \Rightarrow \neg B}{\Gamma \Rightarrow \bot}\ \vee_c E
\qquad
\frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B}\ \vee_c I
$$

$$
\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \bot E
\qquad
\frac{\Gamma, \neg p_i \Rightarrow \bot}{\Gamma \Rightarrow p_c}\ I_c
\qquad
\frac{\Gamma \Rightarrow p_c \quad \Gamma \Rightarrow \neg p_i}{\Gamma \Rightarrow \bot}\ E_c
\qquad
\frac{}{A, \Gamma \Rightarrow A}\ \mathsf{init}
$$

Fig. 1. Ecumenical natural deduction system NEci.

$$
\frac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \to_i B \Rightarrow C}\ \to_i L
\quad
\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to_i B}\ \to_i R
\quad
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow \neg B}{\Gamma, A \to_c B \Rightarrow \bot}\ \to_c L
\quad
\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \to_c B}\ \to_c R
$$

$$
\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \wedge L
\qquad
\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R
\qquad
\frac{\Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow \bot}\ \neg L
\qquad
\frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A}\ \neg R
$$

$$
\frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee_i B \Rightarrow C}\ \vee_i L
\quad
\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee_i A_2}\ \vee_i R
\quad
\frac{\Gamma \Rightarrow \neg A \quad \Gamma \Rightarrow \neg B}{\Gamma, A \vee_c B \Rightarrow \bot}\ \vee_c L
\quad
\frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B}\ \vee_c R
$$

$$
\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \mathsf{W}
\qquad
\frac{\Gamma \Rightarrow \neg p_i}{\Gamma, p_c \Rightarrow \bot}\ L_c
\qquad
\frac{\Gamma, \neg p_i \Rightarrow \bot}{\Gamma \Rightarrow p_c}\ R_c
\qquad
\frac{}{A, \Gamma \Rightarrow A}\ \mathsf{init}
\qquad
\frac{\Gamma \Rightarrow A \quad \Gamma, A \Rightarrow C}{\Gamma \Rightarrow C}\ \mathsf{cut}
$$

Fig. 2. Ecumenical sequent calculus system LEci.

co-exist in $\mathcal{L}$ but have different meanings. We will use a subscript c for the classical meaning and i for the intuitionistic, dropping such subscripts when formulae/connectives can have either meaning. The logical connectives $\{\bot, \neg, \wedge\}$ are common for classical and intuitionistic logic, while $\{\to_i, \vee_i\}$ and $\{\to_c, \vee_c\}$ are restricted to intuitionistic and classical interpretations, respectively.

The natural deduction ecumenical system proposed has been proved strong normalising and sound and complete w.r.t intuitionistic logic's Kripke semantics in [PR17]. In Figure 1 we present NEci, a natural deduction ecumenical system with the more modern sequent presentation. Sequents have the form $\Gamma \Rightarrow C$, with Γ a set of ecumenical formulae and C a formula.

3 The system LEci

It is an easy exercise to transform the natural deduction system NEci into the sequent calculus LEci, depicted in Fig. 2. In fact, as usual the introduction rules become right rules and the elimination rules give rise to the left rules using direct inductive defined translations, for normal derivations or introducing cuts for indirect translations, *e.g.*

$$
\frac{\Gamma \Rightarrow A \vee_c B \quad \Gamma \Rightarrow \neg A \quad \Gamma \Rightarrow \neg B}{\Gamma \Rightarrow \bot}\ \vee_c E
\quad \rightsquigarrow \quad
\frac{\Gamma \Rightarrow \neg A \quad \Gamma \Rightarrow \neg B \quad \overline{\overline{\neg B, \neg A, \Gamma, A \vee_c B \Rightarrow \bot}}}{\Gamma, A \vee_c B \Rightarrow \bot}\ \mathsf{mcut}
\quad \rightsquigarrow \quad
\frac{\Gamma \Rightarrow \neg A \quad \Gamma \Rightarrow \neg B}{\Gamma, A \vee_c B \Rightarrow \bot}\ \vee_c L
$$

The references [vP03, EDH15] bring interesting discussions about isomorphic translations between natural deduction and sequent calculus.

In order to prove cut-elimination for LEci, we use the following ecumenical weight for formulae.

Definition 3.1 The ecumenical weight (ew) of a formula in $\mathcal{L}$ is recursively defined as

- $\mathsf{ew}(p_i) = \mathsf{ew}(\bot) = 0$; $\mathsf{ew}(p_c) = 2$;
- $\mathsf{ew}(A \star B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 1$ if $\star \in \{\wedge, \to_i, \vee_i\}$;
- $\mathsf{ew}(\neg A) = \mathsf{ew}(A) + 1$;
- $\mathsf{ew}(A \circ B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 2$ if $\star \in \{\to_c, \vee_c\}$;

The following are the usual definitions of height of derivations and cut-height [NvP01].

Definition 3.2 The *height of a derivation* is the greatest number of successive applications of rules in it, where an axiom height 0. The *cut-height* of an instance of the cut rule in a derivation is the sum of heights of derivation of the two premisses of cut.

Theorem 3.3 *The system* LEci *has the cut-elimination property.*

Proof. The proof is by the usual Gentzen method, using as inductive measure the pair (n, m), where n is the ecumenical weight of the cut formula and m is the cut-hight of the instance of cut. The principal cases either eliminate the top-most cut or substitute it for cuts over formulae with smaller height *e.g.*

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma, A, \neg B \Rightarrow \bot}}{\Gamma \Rightarrow A \rightarrow_c B}\,\rightarrow_c R \qquad \dfrac{\overset{\pi_2}{\Gamma \Rightarrow A} \quad \overset{\pi_3}{\Gamma \Rightarrow \neg B}}{A \rightarrow_c B, \Gamma \Rightarrow \bot}\,\rightarrow_c L}{\Gamma \Rightarrow \bot}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\pi_2}{\Gamma \Rightarrow A} \qquad \dfrac{\overset{\pi_3}{\Gamma, A \Rightarrow \neg B} \quad \overset{\pi_1}{\Gamma, A, \neg B \Rightarrow \bot}}{\Gamma, A \Rightarrow \bot}\,\mathsf{cut}}{\Gamma \Rightarrow \bot}\,\mathsf{cut}$$

Observe that, in the original cut, $\mathsf{ew}(A \rightarrow_c B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 2$ while the other cuts have associated ecumenical weights $\mathsf{ew}(A)$ and $\mathsf{ew}(B) + 1$, hence being strictly smaller.

The non-principal cuts can be flipped up, generating cuts with smaller cut-height, *e.g.*

$$\dfrac{\overset{\pi_1}{\Gamma \Rightarrow C} \qquad \dfrac{\overset{\pi_2}{C, \Gamma, \neg A, \neg B \Rightarrow \bot}}{C, \Gamma \Rightarrow A \vee_c B}\,\vee_c R}{\Gamma \Rightarrow A \vee_c B}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\pi_1}{\Gamma, \neg A, \neg B \Rightarrow C} \quad \overset{\pi_2}{C, \Gamma, \neg A, \neg B \Rightarrow \bot}}{\Gamma, \neg A, \neg B \Rightarrow \bot}\,\mathsf{cut}}{\Gamma \Rightarrow A \vee_c B}\,\vee_c R$$

We just observe that the classical rules are less permissible, so they allow for more strict applications of the cut rule. In fact, if the cut-formula in the right premise is classical and principal, then $C = \bot$. □

Denoting by $\vdash_{\mathsf{LEci}} A$ the fact that the formula A is a theorem in the system LEci, the following theorems are easily provable

(i) $\vdash_{\mathsf{LEci}} (A \rightarrow_i B) \rightarrow_i (A \rightarrow_c B)$

(ii) $\vdash_{\mathsf{LEci}} (A \wedge B) \leftrightarrow_i \neg(\neg A \vee_c \neg B)$

(iii) $\vdash_{\mathsf{LEci}} (A \wedge B) \leftrightarrow_i \neg(A \rightarrow_c \neg B)$

(iv) $\vdash_{\mathsf{LEci}} \neg(\neg A \wedge \neg B) \leftrightarrow_i (A \vee_c B)$

(v) $\vdash_{\mathsf{LEci}} \neg(A \wedge \neg B) \leftrightarrow_i (A \rightarrow_c B)$.

These theorems are of interest since they relate the classical and the intuitionistic operators. In particular, observe that the intuitionistic implication implies the classical one, but not the opposite, that is $\nvdash_{\mathsf{LEci}} (A \rightarrow_c B) \rightarrow_i (A \rightarrow_i B)$ in general. But a very interesting observation is that $\vdash_{\mathsf{LEci}} (A \rightarrow_c \bot) \leftrightarrow_i (A \rightarrow_i \bot) \leftrightarrow_i (\neg A)$, which means that negation could be defined as $A \rightarrow_i \bot$ or $A \rightarrow_c \bot$ indistinguishably. However, it is interesting to keep the negation operator in the language since the calculus make a heavy use of it.

On the other hand,

(i) $\vdash_{\mathsf{LEci}} A \vee_c \neg A$ but $\nvdash_{\mathsf{LEci}} A \vee_i \neg A$

(ii) $\vdash_{\mathsf{LEci}} (\neg\neg A) \rightarrow_c A$ but $\nvdash_{\mathsf{LEci}} (\neg\neg A) \rightarrow_i A$

(iii) $\vdash_{\mathsf{LEci}} (A \wedge (A \rightarrow_i B)) \rightarrow_i B$

(iv) $\nvdash_{\mathsf{LEci}} (A \wedge (A \rightarrow_c B)) \rightarrow_i B$ in general.

Observe that (iii) means that *modus ponens* is intuitionistically valid, while (iv) implies that it is not classically valid, in general. (iii) has also as a consequence that $A \Rightarrow B$ is provable in LEci iff $\vdash_{\mathsf{LEci}} A \rightarrow_i B$, meaning that *validity* of sequents in *any* semantical interpretation of LEci should correspond to *provability* of the corresponding intuitionistic implicational formula. This corroborates with the results in [PR17] and also brings the new result that the rule $\rightarrow_i R$ is *invertible*.

Definition 3.4 Let $\mathcal{S}$ be a sequent system. An inference rule

$$\dfrac{S_1 \quad \cdots \quad S_n}{S}$$

is called:

i. *admissible in* $\mathcal{S}$ if S is derivable in $\mathcal{S}$ whenever $S_1, \ldots, S_n$ are derivable in $\mathcal{S}$.

$$\frac{\Gamma \Rightarrow A \quad \Gamma, B \Rightarrow \bot}{\Gamma, A \to_c B \Rightarrow \bot} \to_c L \quad \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A} \neg R \quad \frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B} \vee_c R \quad \frac{\Gamma, p_i \Rightarrow \bot}{\Gamma, p_c \Rightarrow \bot} L_c$$

Fig. 3. Some rules for the ecumenical sequent calculus system LEci'.

ii. *invertible* in $\mathcal{S}$ if the rules $\frac{S}{S_1}, \ldots, \frac{S}{S_n}$ are admissible in $\mathcal{S}$.

Proposition 3.5 *The rules $\to_i R, \to_c R, \wedge R, \wedge L, \neg R, \vee_i L, \vee_c R, R_c$ are invertible in* LEci*, while the remaining introduction rules are not.*

Observe that this result allows for a proposal of a better sequent ecumenical system, since the invertibility of the negation right rule implies that they can be decomposed eagerly. Figure 3 presents the system LEci', equivalent to LEci but better from a proof theoretical point of view. In particular, observe that the left rules for implication and disjunction have the same shape now, only differing in the consequent. In the sense that LEci' introduces (bottom up) less negations, one could say that it is "purer" than LEci.

The next step would be proposing a normal form on derivations in LEci'. This is usually done by analysing the *permutability* of rules.

Definition 3.6 In a rule introducing a connective in LEci', the formula with that connective in the conclusion sequent is the *principal formula*, and its sub-formulas ou negated sub-formulas in the premises are the *auxiliary formulas*. Let r_1 and r_2 be inference rules in a sequent system $\mathcal{S}$. The rule r_2 *permutes down* r_1 if for every $\mathcal{S}$-derivation of a sequent S in which r_1 operates on S and r_2 operates on one or more of r_1's premises (but not on auxiliary formulas of r_1), there exists another $\mathcal{S}$-derivation of S in which r_2 operates on S and r_1 operates on zero or more of r_2's premises (but not on auxiliary formulas of r_2).

Theorem 3.7 *The invertible rules in* LEci' *permute down with any other rule. For any other pair of rules, r_i does not permute down with r_c and vice versa, for any intuitionistic non-invertible rule r_i and classical non-invertible rule r_c.*

This means that, for proving a sequent $\Gamma \Rightarrow C$ in LEci' one could apply the invertible rules eagerly and then there is no complete successful proof strategy. This is not entirely surprising, since LJ itself does not have a focused system complete w.r.t. different assignment of polarities for atoms.

However, if the formula is *totally classical*, containing only classical connectives and constants plus $\neg, \wedge$ are considered, then a sort of *goal directed proof search* [MNPS91] can be defined, where right rules should be applied before the left ones.

Finally, as already observed in [PR17], for preserving the "classical behaviour", *i.e.*, satisfying all the principles of classical logic *e.g. modus ponens* and the *classical reductio*, it is sufficient that the main operator of the formulae is classical. Thus, "hybrid" formulae, *i.e.*, with formulae that contain classical and intuitionistic operators may have a classical behaviour. Formally

Definition 3.8 A formula B is called *externally classical* (denoted by B^c) if and only if its main operator is classical.

For externally classical formulae we can now prove the following theorems

(i) $\vdash_{\mathsf{LEci}} (A \to_c B^c) \to_i (A \to_i B^c)$

(ii) $\vdash_{\mathsf{LEci}} (A \wedge_c (A \to_c B^c)) \to_i B^c$

(iii) For every A and B, $C = (A \to_c B)$ and $C = (A \vee_c B)$ satisfy the classical reductio, that is, if $\vdash_{\mathsf{LEci}} (\neg C) \to_i \bot$ then $\vdash_{\mathsf{LEci}} C$.

4 Discussion and future research directions

There is plenty of room for discussion in what was presented in this short paper.

First of all, we believe that it should feasible to extend our reasoning so that to consider quantifiers, although the results for normalisation in the natural deduction setting were restricted to the propositional case (so far).

Second, as we said in the introduction, the idea of using different signs for the different meanings attached to intuitionistic and classical operators is not new. Hence, there is a lot of comparison to be done with other systems in the literature that combine intuitionistic and classical logics. Most importantly, there is the seminal work of Girard in [Gir93], that was somehow subsumed by the work of Liang and Miller [LM11]. These works are based on *polarities*, the last using also some translations into linear logic, so it may be the case that there is some intersection, but not that much. Indeed, the proof search space in those systems seem to be completely different from ours. Also, Kraus [Kra92] in 1992 and Dowek [Dow16] in 2015 proposed systems that have a single negation, but they both have classical versions of $\wedge$ and $\forall$. It is interesting to observe that (1) $\wedge_c$ does not satisfy (in general) projections and is not idempotent and that (2) $\forall_c$ does not (in general) satisfy universal instantiation. The main motivation of Krauss and Dowek was to explore the possibility of *hybrid* readings of axioms and proofs in mathematical theories. They consider examples taken from set theory: Krauss explores different readings of the axiom of choice and Dowek considers a *ecumenical* proof of the theorem that asserts that if the union of two sets is infinite, then at least one of the two sets is infinite. The whole point is, in Dowek's own words, to consider that "which mathematical results have a classical formulation that can be proved from the axioms of constructive set theory or constructive type theory and which require a classical formulation of these axioms and a classical notion of entailment remais to be investigated".

Regarding typing, we hope that our investigation will allow us to provide systems of terms for the Ecumenical system, thereby explaining more thoroughly the differences between the several kinds of classical logic Curry-Howard correspondences in the literature.

Finally, a world about semantics. Prawitz' initial motivation about ecumenism was in the direction of having a good proof theoretical semantics for classical logic. In essence, it would be like the intuitionistic world trying to give some meaning to what the classical system accepts as true. This is, in fact, what the double negation translation does. We showed that *any* semantical interpretation of sequents in LEci is, in fact, intuitionistic. Hence it seems that Prawitz is right, as always.

References

[Dow16] Gilles Dowek. On the definition of the classical connectives and quantifiers. *Why is this a Proof?, Festschrift for Luiz Carlos Pereira*, 27:228–238, 2016.

[dPP05] Valeria de Paiva and Luiz Carlos Pereira. A short note on intuitionistic propositional logic with multiple conclusions. *Manuscrito*, 28(2):317–329, 2005.

[EDH15] Cécilia Englander, Gilles Dowek, and Edward Hermann Haeusler. Yet another bijection between sequent calculus and natural deduction. *Electr. Notes Theor. Comput. Sci.*, 312:107–124, 2015.

[Fit14] Melvin Fitting. Nested sequents for intuitionistic logics. *Notre Dame Journal of Formal Logic*, 55(1):41–61, 2014.

[Gen69] Gerhard Gentzen. *The Collected Papers of Gerhard Gentzen*. Amsterdam: North-Holland Pub. Co., 1969.

[Gir87] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[Gir93] Jean-Yves Girard. On the unity of logic. *Ann. Pure Appl. Logic*, 59(3):201–217, 1993.

[How80] William A. Howard. The formulae-as-type notion of construction, 1969. In J. P. Seldin and R. Hindley, editors, *To H. B. Curry: Essays in Combinatory Logic, Lambda Calculus, and Formalism*, pages 479–490. Academic Press, New York, 1980.

[Kra92] Peter Krauss. A constructive refinement of classical logic. Draft, 1992.

[LM11] Chuck Liang and Dale Miller. A focused approach to combining logics. *Ann. Pure Appl. Logic*, 162(9):679–697, 2011.

[LPR18] Björn Lellmann, Elaine Pimentel, and Revantha Ramanyake. Sequentialising nested systems. draft, 2018.

[Mae54] S. Maehara. Eine darstellung der intuitionistischen logik in der klassischen. *Nagoya Mathematical Journal*, pages 45–64, 1954.

[MNPS91] Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51:125–157, 1991.

[NvP01] Sara Negri and Jan von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.

[Par92] Michel Parigot. $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction. In *LPAR: Logic Programming and Automated Reasoning, International Conference*, volume 624 of *LNCS*, pages 190–201. Springer, 1992.

[PR17] Luiz Carlos Pereira and Ricardo Oscar Rodriguez. Normalization, soundness and completeness for the propositional fragment of prawitz ecumenical system. *Revista Portuguesa de Filosofia*, 73(3-3):1153–1168, 2017.

[Pra65] Dag Prawitz. *Natural Deduction, volume 3 of Stockholm Studies in Philosophy*. Almqvist and Wiksell, 1965.

[Pra16] Dag Prawitz. Classical versus intuitionistic logic. *Why is this a Proof?, Festschrift for Luiz Carlos Pereira*, 27:15–32, 2016.

[Vig00] Luca Viganò. *Labelled non-classical logics*. Kluwer, 2000.

[vP03] Jan von Plato. Translations from natural deduction to sequent calculus. *Math. Log. Q.*, 49(5):435–443, 2003.