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# CATEGORIFYING COMPUTABLE REDUCIBILITIES

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**ABSTRACT.** This paper discusses categorical formulations of the Medvedev, Muchnik and Weihrauch reducibilities in Computability Theory and connects these reducibilities to different Lawvere categorical doctrines. These specific doctrines were used in previous work to provide categorical models for Dialectica logical properties. We relate Medvedev and Weihrauch doctrines to the Dialectica doctrine, showing that all these doctrines can be conceptualized in terms of (logic) quantifier (existential and universal) completions.

## 1. INTRODUCTION

Categorical methods and language have been employed in many areas of Mathematics. In Mathematical Logic they are widely used in Model Theory and Proof Theory, less so in Set Theory. After all, category theory is sometimes seen as a competitor to Set Theory, as both are considered foundations for mathematics. In Recursion or Computability Theory, there is a long tradition of categorical methods in Realizability studies, expounded in the van Oosten book “Realizability: An Introduction to its Categorical Side” [vO08]. The use of categorical methods in Realizability includes, for example, Hofstra’s work on “All Realizability is Relative” [Hof06]. Hofstra proved that most well-known realizability triposes, (e.g. the “effective” tripos, the “modified realizability” tripos and the “dialectica” tripos) are instances of a more general notion of tripos associated to a given partial combinatory algebra (PCA). His point was that all these triposes can be presented as “triposes for a given PCA”, hence all these notions differ only in the choice of the associated PCA, and we could say that all realizability is relative to a choice of a PCA. This formulation explains and systematizes the different kinds of triposes for realizability developed independently before.

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Previous work of Trotta and Maietti [MT23] also showed that every realizability tripos (in the general sense of a tripos constructed from a PCA) is an instance of the generalized **existential completion**. So every realizability tripos is obtained by freely adding left adjoints along the class of all morphisms of the base category. Hence, in particular, the dialectica tripos (Biering in [Bie08], the tripos of modified realizability, etc... are all instances of the generalized existential completion. This reformulation explains and systematizes the different kinds of triposes for realizability in terms of an existential completion.

In this paper we want to extend the use of categorical methods to further areas of Computability. We want to discuss categorical formulations of Medvedev, Muchnik and Weihrauch reducibility [Hin12] and connect these to different categorical doctrines used in our previous work [TSdP22, TSdP21]. In particular, we relate Medvedev and Weihrauch doctrines to the Dialectica doctrine, showing that all these doctrines can be conceptualized in terms of quantifier completions.

Putting together a categorical understanding of reducibility in computability with categorical logic semantic descriptions in terms of ‘(hyper)doctrines’ to show a bridge between these two areas of mathematical logic seems helpful to both sides. For categorical proof-theorists it encompasses models of both functional interpretations and computability under the same kind of construction: both realizability/Dialectica triposes and reducibility degrees are versions of quantifier completions. For recursion theorists it extends the categorification of realizability (described e.g. in [vO08]) to degrees of reducibility (Medvedev, Muchnik and Weihrauch) not usually considered in categorical terms. The categorical formulation allows for abstract, easier proofs of results.

The outline of this paper is as follows. In section 2 we recall general computability definitions we need. In section 3 we recall doctrines and quantifier completions. In section 4 we reformulate Medvedev and Muchnik reducibility in categorical terms. In Section 5 we categorify Weihrauch reducibility and in Section 6 we compare Medvedev and Weihrauch doctrines to Dialectica ones. We draw some general lessons in section 7.

## 2. COMPUTABILITY NOTIONS

Realizability theory originated with Kleene’s interpretation of intuitionistic number theory [Kle45] and has since developed into a large body of work in logic and theoretical computer science. We focus on two basic flavours of realizability, number realizability and function realizability, which were both due to Kleene.

In this section we recall some standard notions within realizability and computability, including partial combinatory algebras, represented spaces, assemblies and modest sets. We follow the approach suggested by Hofstra [Hof04], as we want to fix a suitable notation for both category theorists and computability logicians.

We describe *partial combinatory algebras* and discuss some important examples. For more details, we refer the reader to van Oosten’s work on categorical realizability [vO08, vO13]. But start by introducing the basic concept of a *partial applicative structure* or PAS, due to Feferman, which may be viewed as a universe for computation.

**Definition 2.1** (PAS). A **partial applicative structure**, or PAS for short, is a set  $\mathbb{A}$  equipped with a **partial** binary operation  $\cdot : \subseteq \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$ .

Some conventions and terminology: Given two elements  $a, b$  in  $\mathbb{A}$ , we think of  $a \cdot b$  (which we often abbreviate  $ab$ ) as “ $a$  applied to  $b$ ”. The partiality of the operation  $\cdot$  means

that this application need not always be defined. We write  $f : \subseteq A \rightarrow B$  to say that  $f$  is a partial function with domain a subset of  $A$ ,  $\text{dom}(f) \subset A$  and range exactly  $\text{ran}(f) = B$ . If  $(a, b) \in \text{dom}(\cdot)$ , that is, when the application is defined, then we write  $a \cdot b \downarrow$  or  $ab \downarrow$ .

We usually omit brackets, assuming associativity of application to the left. Thus  $abc$  stands for  $(ab)c$ . Moreover, for two expressions  $x$  and  $y$  we write  $x \simeq y$  to indicate that  $x$  is defined whenever  $y$  is, in which case they are equal.

Even though these partial applicative structures do not possess many interesting properties (they have no axioms for application), they already highlight one of the key features of combinatorial structures, namely the fact that we have a domain of elements that can act both as functions and as arguments, just as in untyped  $\lambda$ -calculus. This behaviour can be traced back to Von Neumann's idea that programs (functions, operations) live in the same realm and are represented in the same way as the data (arguments) that they act upon. In particular, programs can act on other programs.

Now we can state the definition of a *partial combinatory algebra* (PCA).

**Definition 2.2** (PCA). A **partial combinatory algebra** (PCA) is a PAS  $\mathbb{A}$  for which there exist elements  $k, s \in \mathbb{A}$  such that for all  $a, b, c \in \mathbb{A}$  we have that

$$ka \downarrow \text{ and } kab \simeq a$$

and

$$sa \downarrow, sab \downarrow, \text{ and } abc \simeq ac(bc)$$

The elements  $k$  and  $s$  are generalisations of the homonymous combinators in Combinatory Logic. Note that appropriate elements  $k, s$  are not considered part of the structure of the partial combinatory algebra, so they do not need to be preserved under homomorphisms.

Given a PCA  $\mathbb{A}$  one can prove its combinatory completeness in the sense of [Fef75, vO08], namely: for every term  $t(x_1, \dots, x_{n+1})$  built from variables  $x_1, \dots, x_{n+1}$ , constants  $\bar{c}$  for  $c \in \mathbb{A}$ , and an application operator  $\cdot$ , there is an element  $a \in \mathbb{A}$  such that for all elements  $b_1, \dots, b_{n+1} \in \mathbb{A}$  we have that  $ab_1 \cdots b_n \downarrow$  and  $ab_1 \cdots b_{n+1} \simeq t(b_1, \dots, b_{n+1})$ .

In particular, we can use this result and the elements  $k$  and  $s$  to construct elements  $p, p_1, p_2$  of  $\mathbb{A}$  so that  $(a, b) \mapsto pab$  is an injection of  $\mathbb{A} \times \mathbb{A}$  to  $\mathbb{A}$  with left inverse  $a \mapsto (p_1a, p_2a)$ . Hence, we can use  $pab$  as an element of  $\mathbb{A}$  which codes the pair  $(a, b)$ . For this reason, the elements  $p, p_1, p_2$  are usually called *pairing* and *projection* operators. For the sake of readability, we write  $\langle a, b \rangle$  in place of  $pab$ .

Using  $s$  and  $k$ , we can prove the analogues of the Universal Turing Machine (UTM) and the SMN theorems in computability in an arbitrary PCA.

A PCA  $\mathbb{A}$  is called **extensional** if for all  $x$  in  $\mathbb{A}$ ,  $(ax \simeq bx)$  implies  $a = b$ . By definition, in every extensional PCA, if elements represent the same partial function, then they must be equal.

**Example 2.3** (*Kleene's first model*). Fix an effective enumeration  $(\varphi_a)_{a \in \mathbb{N}}$  of the partial recursive functions  $\mathbb{N} \rightarrow \mathbb{N}$  (i.e. a Gödel numbering). The set  $\mathbb{N}$  with partial recursive application  $(a, b) \mapsto \varphi_a(b)$  is a PCA, and it is called *Kleene's first model*  $\mathcal{K}_1$  (see e.g. [Soa87]).

**Example 2.4** (*Kleene's second model*).  $\mathcal{K}_2$  is a PCA for function realizability [vO08, Sec. 1.4.3]. This PCA is given by the Baire space  $\mathbb{N}^{\mathbb{N}}$ , endowed with the product topology. The partial binary operation of application  $\cdot : \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  corresponds to the one used in Type-2 Theory of Effectivity [Wei00]. This can be described as follows. Let  $\alpha[n]$  denote

the string  $(\alpha(0), \dots, \alpha(n-1))$ . Every  $\alpha \in \mathbb{N}^{\mathbb{N}}$  induces a function  $F_\alpha : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$  defined as  $F_\alpha(\beta) = k$  if there is  $n \in \mathbb{N}$  such that  $\alpha(\langle \beta[n] \rangle) = k+1$  and  $(\forall m < n)(\alpha(\langle \beta[m] \rangle) = 0)$ , and undefined otherwise. The application  $\alpha \cdot \beta$  can then be defined as the map  $n \mapsto F_\alpha((n)^\wedge \beta)$ , where  $(n)^\wedge \beta$  is the string  $\sigma$  defined as  $\sigma(0) := n$  and  $\sigma(k+1) := \beta(k)$ .

**Remark 2.5.** Kleene's  $\mathcal{K}_1$  and  $\mathcal{K}_2$  presented above are not extensional, since there are many codes (programs) that compute the same function. In fact, every function has infinitely many representatives.

Next we recall the notion of *elementary sub-PCA*. Many definitions in the computability context refer to a concept and a subset of the given concept, as one needs to pay attention to the computable functions (and elements) included in the original concept.

**Definition 2.6** (elementary sub-PCA). Let  $\mathbb{A}$  be a PCA. A subset  $\mathbb{A}' \subseteq \mathbb{A}$  is called an **elementary sub-PCA** of  $\mathbb{A}$  if it is closed under the application of  $\mathbb{A}$ , that means: if  $a, b \in \mathbb{A}'$  and  $ab \downarrow$  in  $\mathbb{A}$  then  $ab \in \mathbb{A}'$  and  $\mathbb{A}'$  is a PCA with the partial applicative structure induced by  $\mathbb{A}$ .

In particular, elements of the sub-PCA will play the role of the computable functions. For example, in Kleene's second model, we usually consider the elementary sub-PCA  $\mathbb{N}_{\text{eff}}^{\mathbb{N}}$  consisting of all the  $\alpha \in \mathbb{N}^{\mathbb{N}}$  such that  $\beta \mapsto \alpha \cdot \beta$  is computable or effective. For more details and examples, also of non-elementary sub-PCAs for Kleene's second model, we refer to [vO11].

We now briefly recall a few definitions in the context of Computable Analysis. For a more thorough presentation, the reader is referred to [BGP21, Wei00].

**Definition 2.7** (represented space). A **represented space** is a pair  $(X, \delta_X)$ , where  $X$  is a set and  $\delta_X : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$  is a surjective map usually called a **representation map**.

We avoid mentioning explicitly the representation map whenever there is no ambiguity. For any given  $x$  in  $X$ , the set  $\delta_X^{-1}(x)$  is the set of  $\delta_X$ -names or  $\delta_X$ -codes for  $x$ .

A (partial) multi-valued function from  $X$  to  $Y$ , written as  $f : \subseteq X \rightrightarrows Y$ , is a function into the powerset  $f : X \rightarrow \wp(Y)$ . The domain of  $f$  is the set  $\{x \in X : f(x) \neq \emptyset\}$ . Whenever  $f(x)$  is a singleton for every  $x \in \text{dom}(f)$ , we write  $f(x) = y$  instead of  $f(x) = \{y\}$ . It helps the intuition to think of multi-valued functions as computational problems, namely instance-solution pairs, where a single problem instance can have multiple solutions.

**Definition 2.8** (realizer). Let  $f : \subseteq X \rightrightarrows Y$  be a multi-valued function between the represented spaces  $(X, \delta_X)$  and  $(Y, \delta_Y)$ . A **realizer**  $F$  for  $f$  (we write  $F \vdash f$ ) is a function  $F : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  such that for all  $p$  in the domain of  $(f \circ \delta_X)$  we have that  $(\delta_Y(F(p)) \in f(\delta_X(p)))$ .

Realizers are useful, they help us transfer properties of functions on the Baire space (such as computability or continuity) to multi-valued functions on the represented spaces.

Because the representing maps in represented spaces are partial, the more convenient category-theoretical notion corresponding to a represented space is the notion of an *assembly* (see [vO08, Def. 1.5.1]). More precisely:

**Definition 2.9** (assembly). Let  $\mathbb{A}$  be a PCA. An **assembly** is a pair  $(X, E)$  where  $X$  is a set and  $E$  is a function  $E : X \rightarrow \wp^*(\mathbb{A})$ , where  $\wp^*(\mathbb{A})$  the set of non-empty subsets of  $\mathbb{A}$ . A morphism of assemblies  $f : (X, E) \rightarrow (Y, F)$  is a function  $f : X \rightarrow Y$  such that there exists an element  $a \in \mathbb{A}$  with the property that for every  $x \in X$  and every  $b \in E(x)$ ,  $ab \downarrow$  and  $ab \in F(f(x))$ . We sometimes say that  $a$  **tracks** or **realizes** the function  $f$ .

Assemblies on a PCA  $\mathbb{A}$  and their morphisms form a category denoted by  $\mathbf{Asm}(\mathbb{A})$ . (Note that the  $E$ s are total functions, even if multi-valued.)

**Definition 2.10** (relative assembly). Let  $\mathbb{A}$  be a PCA and  $\mathbb{A}' \subset \mathbb{A}$  be an elementary sub-PCA of  $\mathbb{A}$ . We define the category of **relative assemblies**  $\mathbf{Asm}(\mathbb{A}, \mathbb{A}')$  as the subcategory of  $\mathbf{Asm}(\mathbb{A})$  having assemblies as objects and where morphisms are morphisms of assemblies with the property that functions have a realizer in  $\mathbb{A}'$ .

**Remark 2.11.** Notice that if  $(X, \delta_X)$  is a represented space according to Definition 2.7, then  $(X, \delta_X^{-1})$  is an assembly as in Definition 2.9.

We can check directly that while every represented space  $(X, \delta_X)$  as in Definition 2.7 gives rise to a unique assembly  $(X, \delta_X^{-1})$  as in Definition 2.9, the converse does not hold in general. Assemblies corresponding to represented spaces are usually called *modest sets* [Ros90].

**Definition 2.12** (modest set). Let  $\mathbb{A}$  be a PCA with elementary sub-PCA  $\mathbb{A}'$ . An assembly  $(X, E)$  is called a **modest set** if for every  $x, y \in X$ , we have  $E(x) \cap E(y) = \emptyset$ . The category of **modest sets** is defined as the full subcategory  $\mathbf{Mod}(\mathbb{A}, \mathbb{A}')$  of  $\mathbf{Asm}(\mathbb{A}, \mathbb{A}')$  whose objects are modest sets.

Since for each modest set there is a corresponding represented space we will make no distinction between these in terms of notation. In particular, we will always adopt the notation of representable spaces  $(X, \delta_X)$  to indicate modest sets.

### 3. CATEGORICAL DOCTRINES

We want to connect the notions of computability described in the previous subsection to work on (logical) categorical doctrines in [TSdP22]. We recap only the essential definitions from the doctrines work in the text; further details and explanation can be found in [TSdP21, TSdP22].

Several generalisations of the notion of a (Lawvere) hyperdoctrine have been considered recently, we refer, for example, to the works of Rosolini and Maietti for that [MPR17, MR13a, MR13b] or to [Pit02, HJP80] for higher-order versions. However, in this work we consider a natural generalisation of the notion of hyperdoctrine, which we call simply a *doctrine*.

**Definition 3.1** (doctrine). A **doctrine** is a contravariant functor:

$$P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Pos}$$

where the category  $\mathcal{C}$  has finite products and  $\mathbf{Pos}$  is the category of (partially ordered sets or) posets.

**Definition 3.2** (morphism of doctrines). A morphism of doctrines is a pair  $\mathcal{L} := (F, \mathfrak{b})$

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \downarrow F^{\text{op}} & \searrow P & \\
 \mathcal{D}^{\text{op}} & & \mathbf{Pos} \\
 & \nearrow R & \\
 & \downarrow \mathfrak{b} & 
 \end{array}$$

such that  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a finite product preserving functor and:

$$\mathfrak{b}: P \rightarrow RF^{\text{op}}$$

is a natural transformation.

**Example 3.3.** Let  $\mathbb{A}$  be a PCA. We can define a functor  $\mathbb{A}^{(-)}: \text{Set}^{\text{op}} \longrightarrow \text{Pos}$  assigning to a set  $X$  the set  $\mathbb{A}^X$  of functions from  $X$  to  $\mathbb{A}$ . Given two elements  $\alpha, \beta \in \mathbb{A}^X$ , we have that  $\alpha \leq \beta$  if there exists an element  $a \in \mathbb{A}$  such that for every  $x \in X$  we have that  $a \cdot \alpha(x)$  is defined and  $a \cdot \alpha(x) = \beta(x)$ . This is the **doctrine associated to the PCA**  $\mathbb{A}$ .

We recall the following example from [Pit02, HJP80].

**Example 3.4.** Given a PCA  $\mathbb{A}$ , we can consider the **realizability doctrine**  $\mathcal{R}: \text{Set}^{\text{op}} \longrightarrow \text{Pos}$  over  $\text{Set}$ . For each set  $X$ , the partial order  $(\mathcal{R}(X), \leq)$  is defined as the set of functions  $\wp(\mathbb{A})^X$  from  $X$  to the powerset  $\wp(\mathbb{A})$  of  $\mathbb{A}$ . Given two elements  $\alpha$  and  $\beta$  of  $\mathcal{R}(X)$ , we say that  $\alpha \leq \beta$  if there exists an element  $\bar{a} \in \mathbb{A}$  such that for all  $x \in X$  and all  $a \in \alpha(x)$ ,  $\bar{a} \cdot a$  is defined and it is an element of  $\beta(x)$ . By standard properties of PCAs this relation is reflexive and transitive, i.e. it is a preorder. Then  $\mathcal{R}(X)$  is defined as the quotient of  $\wp(\mathbb{A})^X$  by the equivalence relation generated by the  $\leq$ . The partial order on the equivalence classes  $[\alpha]$  is the one induced by  $\leq$ .

We also need to recall the definitions of existential and universal doctrines, in general.

**Definition 3.5** ( $\mathcal{D}$ -existential/universal doctrines). Let  $\mathcal{C}$  be a category with finite products, and let  $\mathcal{D}$  a class of morphisms of  $\mathcal{C}$  closed under composition, pullbacks and identities. A doctrine  $P: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  is  **$\mathcal{D}$ -existential** (resp.  **$\mathcal{D}$ -universal**) if, for every arrow  $f: A \rightarrow B$  of  $\mathcal{D}$  the functor:

$$P_f: PB \rightarrow PA$$

has a left adjoint  $\exists_f$  (resp. a right adjoint  $\forall_f$ ), and these satisfy the Beck-Chevalley condition BC: for any pullback diagram:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{g} & A \end{array}$$

and any  $\beta$  in  $P(X)$  the equality:

$$\exists_{g'} P_{h'} \beta = P_h \exists_g \beta \quad (\text{resp. } \forall_{g'} P_{h'} \beta = P_h \forall_g \beta)$$

holds. When  $\mathcal{D}$  is the class of all the morphisms of  $\mathcal{C}$  we say that the doctrines  $P$  is **full existential** (resp. **full universal**), while when  $\mathcal{D}$  is the class of product projections, we will say that  $P$  is **pure existential** (resp. **pure universal**).

Next we summarise the main properties of the generic full existential and universal completions in the following theorems and refer to [Tro20] for more details.

**Generalized existential completion.** Let  $P: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  be a doctrine and let  $\mathcal{D}$  be a class of morphisms of  $\mathcal{C}$  closed under composition, pullbacks and containing identities. For every object  $A$  of  $\mathcal{C}$  consider the following preorder:

- **objects:** pairs  $(B \xrightarrow{f \in \mathcal{D}} A, \alpha)$ , where  $f: B \rightarrow A$  is an arrow of  $\mathcal{D}$  and  $\alpha \in P(B)$ .

- **order:**  $(B \xrightarrow{f \in \mathcal{D}} A, \alpha) \leq (C \xrightarrow{g \in \mathcal{D}} A, \beta)$  if there exists an arrow  $h: B \longrightarrow C$  of  $\mathcal{C}$  such that the diagram

$$\begin{array}{ccc} & & B \\ & \swarrow h & \downarrow f \\ C & \xrightarrow{g} & A \end{array}$$

commutes and

$$\alpha \leq P_h(\beta)$$

It is easy to see that the previous data gives a preorder. We denote by  $P^{\exists \mathcal{D}}(A)$  the partial order obtained by identifying two objects when

$$(B \xrightarrow{h \in \mathcal{D}} A, \alpha) \gtrsim (D \xrightarrow{f \in \mathcal{D}} A, \gamma)$$

in the usual way. By abuse of notation, we denote the equivalence class of an element in the same way. Given a morphism  $f: A \rightarrow B$  in  $\mathcal{C}$ , let  $P_f^{\exists \mathcal{D}}(C \xrightarrow{g} B, \beta)$  be the object

$$(D \xrightarrow{f^*g \in \mathcal{D}} A, P_{g^*f}(\beta))$$

where  $f^*g$  and  $g^*f$  are defined by the pullback

$$\begin{array}{ccc} D & \xrightarrow{g^*f} & C \\ \downarrow f^*g & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & B. \end{array}$$

The assignment  $P^{\exists \mathcal{D}}: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  is called the **generalized existential completion** of  $P$ . Following [Tro20, MT23], when  $\mathcal{D}$  is the class of all the morphisms of the base category, we will speak of **full existential completion**, and we will use the notation  $P^{\exists \mathcal{I}}: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$ . Moreover, when  $\mathcal{D}$  is the class of all product projections, we will speak of **pure existential completion**, and we will use the notation  $P^{\exists}: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$ .

**Theorem 3.6** ([Tro20]). *The doctrine  $P^{\exists \mathcal{D}}$  is  $\mathcal{D}$ -existential. Moreover, for every doctrine  $P$  we have a canonical inclusion  $\eta_P^{\exists \mathcal{D}}: P \rightarrow P^{\exists \mathcal{D}}$  such that, for every morphism of doctrines  $\mathcal{L}: P \rightarrow R$ , where  $R$  is  $\mathcal{D}'$ -existential and the functor between the bases sends arrows of  $\mathcal{D}$  into arrows of  $\mathcal{D}'$ , there exists a unique (up to iso) existential morphism doctrine (i.e. preserving existential quantifiers along  $\mathcal{D}$ ) such that the diagram*

$$\begin{array}{ccc} & & P^{\exists \mathcal{D}} \\ & \nearrow \eta_P^{\exists \mathcal{D}} & \downarrow \dashv \\ P & \xrightarrow{\mathcal{L}} & R \end{array}$$

commutes.

By dualising the previous construction, we can define the  $\mathcal{D}$ -universal completion of a doctrine.

**Full universal completion.** Let  $P: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  be a doctrine and let  $\mathcal{D}$  be a class of morphisms of  $\mathcal{C}$  closed under composition, pullbacks and containing identities. For every object  $A$  of  $\mathcal{C}$  consider the following preorder:

- **objects:** pairs  $(B \xrightarrow{f \in \mathcal{D}} A, \alpha)$ , where  $f: B \rightarrow A$  is an arrow of  $\mathcal{D}$  and  $\alpha \in P(B)$ .
- **order:**  $(B \xrightarrow{f \in \mathcal{D}} A, \alpha) \leq (C \xrightarrow{g \in \mathcal{D}} A, \beta)$  if there exists an arrow  $h: C \longrightarrow B$  of  $\mathcal{C}$  such that the diagram

$$\begin{array}{ccc} & B & \\ & \nearrow h & \downarrow f \\ C & \xrightarrow{g} & A \end{array}$$

commutes and

$$P_h(\alpha) \leq \beta$$

Again, it is easy to see that the previous data gives us a preorder. We denote by  $P^{\forall \mathcal{D}}(A)$  the partial order obtained by identifying two objects when

$$(B \xrightarrow{h \in \mathcal{D}} A, \alpha) \cong (D \xrightarrow{f \in \mathcal{D}} A, \gamma)$$

in the usual way. By abuse of notation, we denote the equivalence class of an element in the same way. Given a morphism  $f: A \rightarrow B$  in  $\mathcal{C}$ , let  $P_f^{\forall \mathcal{D}}(C \xrightarrow{g \in \mathcal{D}} B, \beta)$  be the object

$$(D \xrightarrow{f^*g \in \mathcal{D}} A, P_{g^*f}(\beta))$$

where  $f^*g$  and  $g^*f$  are defined by the pullback

$$\begin{array}{ccc} D & \xrightarrow{g^*f} & C \\ \downarrow f^*g & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

The assignment  $P^{\forall \mathcal{D}}: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  is called the  $\mathcal{D}$ -universal completion of  $P$ . As before, when  $\mathcal{D}$  is the class of all the morphisms of the base category, we will speak of **full universal completion**, and we will use the notation  $P^{\forall \mathcal{D}}: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$ . Moreover, when  $\mathcal{D}$  is the class of all product projections, we will speak of **pure universal completion**, and we will use the notation  $P^{\forall}: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$ .

**Theorem 3.7** ([Tro20]). *The doctrine  $P^{\forall \mathcal{D}}$  is  $\mathcal{D}$ -universal. Moreover, for every doctrine  $P$  we have a canonical inclusion  $\eta_P^{\forall \mathcal{D}}: P \rightarrow P^{\forall \mathcal{D}}$  such that, for every morphism of doctrines  $\mathcal{L}: P \rightarrow R$ , where  $R$  is  $\mathcal{D}'$ -universal and the functor between the bases sends arrows of  $\mathcal{D}$*

into arrows of  $\mathcal{D}'$ , there exists a unique (up to isomorphism) universal morphism doctrine (i.e. preserving universal quantifiers along  $\mathcal{D}$ ) such that the diagram

$$\begin{array}{ccc}
 & & P^{\forall \mathcal{D}} \\
 & \nearrow \eta_P^{\forall \mathcal{D}} & | \\
 P & \xrightarrow{\varepsilon} & R \\
 & & \downarrow \forall \\
 & & R
 \end{array}$$

commutes.

Now that we recalled both basic concepts of computability and the tools we need from Lawvere doctrines we can start on the computability concepts we want to categorify here. First we recall Medvedev reducibility and show it can be reformulated as a Medvedev doctrine.

#### 4. MEDVEDEV DOCTRINES

The notion of Medvedev reducibility was introduced in the 50s, associated to a calculus of mathematical problems<sup>1</sup> in the style of Kolmogorov [Kol91], and now it is well established in the computability literature. We briefly introduce the main notions and definitions on the topic. For a more thorough presentation, the reader is referred to [Sor96, Hin12].

A set  $A \subseteq \mathbb{N}^{\mathbb{N}}$  is sometimes called a *mass problem*. The intuition is that a mass problem corresponds to the set of solutions for a specific computational problem. For example, the problem of deciding membership in a particular  $P \subseteq \mathbb{N}$  corresponds to the mass problem  $\{\chi_P\}$ , where  $\chi_P$  is the characteristic function of  $P$ . Similarly, the problem of enumerating  $P$  corresponds to the family  $\{f: \mathbb{N} \rightarrow P \text{ such that } f \text{ is surjective}\}$ .

While Medvedev reducibility is usually defined in the context of Type-2 computability, we can give a slightly more general definition in the context of PCAs.

**Definition 4.1** (Medvedev reducible set). Let  $\mathbb{A}$  be a PCA and let  $\mathbb{A}'$  be an elementary sub-PCA of  $\mathbb{A}$ . If  $A, B \subseteq \mathbb{A}$ , we say that set  $A$  is **Medvedev reducible** to set  $B$ , and write  $A \leq_M B$ , if there is an effective functional  $\Phi \in \mathbb{A}'$  such that  $\Phi(B) \subseteq A$ , i.e.  $(\forall b \in B)(\Phi(b) \in A)$ .

The notion of Medvedev reducibility induces a quasi-order on the powerset of  $\mathbb{A}$ ,  $\wp(\mathbb{A})$ , whose quotient is the *Medvedev lattice*. In the following, whenever there is no ambiguity, we identify a degree with any of its representatives.

The quotient of the Medvedev order is the Medvedev lattice. The Medvedev doctrine maps every singleton to an isomorphic copy of the Medvedev lattice. However, if  $X$  is not a singleton, we obtain a somewhat different structure, as mentioned after the definition. Intuitively, it is like having several Medvedev reductions all witnessed by the same map.

We are ready to start defining a Medvedev doctrine. We first define a *Medvedev doctrine of singletons*.

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<sup>1</sup>Recently de Paiva and Da Silva showed that Kolmogorov problems can be seen as a variant of the Dialectica construction [dPdS21].

**Definition 4.2.** Let  $\mathbb{A}$  be a PCA and  $\mathbb{A}'$  be an elementary sub-PCA. We can define a functor  $\mathbb{A}_M: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  mapping a set  $X$  to the set  $\mathbb{A}^X$  of functions from  $X$  to  $\mathbb{A}$ . Given two elements  $\alpha, \beta \in \mathbb{A}_M(X)$ , we define  $\alpha \leq_{\text{dM}} \beta$  if there exists an element  $\bar{a} \in \mathbb{A}'$  such that for every  $x \in X$  we have that  $\bar{a} \cdot \beta(x)$  is defined and  $\bar{a} \cdot \beta(x) = \alpha(x)$ . The functor  $\mathbb{A}_M: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  is called **Medvedev doctrine of singletons**.

**Definition 4.3** (Medvedev doctrine). Given a PCA  $\mathbb{A}$  with elementary sub-PCA  $\mathbb{A}'$ , we define the **Medvedev doctrine**  $\mathfrak{M}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  over  $\mathbf{Set}$  as follows: for every set  $X$  and every pair of functions  $\varphi, \psi$  in  $\mathcal{P}(\mathbb{A})^X$ , we define

$$\begin{aligned} \varphi \leq_{\text{M}} \psi &: \iff (\exists \bar{a} \in \mathbb{A}')(\forall x \in X)(\forall b \in \psi(x))(\exists a \in \varphi(x))(\bar{a} \cdot b = a) \\ &\iff (\exists \bar{a} \in \mathbb{A}')(\forall x \in X)(\bar{a} \cdot \psi(x) \subseteq \varphi(x)). \end{aligned}$$

This preorder induces an equivalence relation on functions in  $\mathcal{P}(\mathbb{A})^X$ . The poset  $\mathfrak{M}(X)$  is defined as the quotient of  $\mathcal{P}(\mathbb{A})^X$  by the equivalence relation generated by the  $\leq_{\text{M}}$ . The partial order on the equivalence classes  $[\varphi]$  is the one induced by the Medvedev order  $\leq_{\text{M}}$ . Moreover, given a function  $f: X \rightarrow Y$ , the functor  $\mathfrak{M}_f: \mathfrak{M}(Y) \rightarrow \mathfrak{M}(X)$  is defined as  $\mathfrak{M}_f(\psi) := \psi \circ f$ .

Observe that, whenever  $X$  is a singleton and  $(\mathbb{A}, \mathbb{A}') = (\mathbb{N}^{\mathbb{N}}, \mathbb{N}_{\text{eff}}^{\mathbb{N}})$  is Kleene's second PCA, the partial order  $\mathfrak{M}(X)$  corresponds exactly to the Medvedev degrees. For a generic  $X \subseteq \mathbb{N}^{\mathbb{N}}$  and  $\varphi, \psi \in \mathfrak{M}(X)$ , the reduction  $\varphi \leq_{\text{M}} \psi$  corresponds to a uniform strong omniscient computable reduction, where the strong omniscient computable reducibility was introduced in [MP19].

Now we want to show that, for every  $X$ ,  $\mathfrak{M}(X)$  is a distributive lattice with  $\perp = \{\mathbb{A}\}$  and  $\top = \emptyset$ , where the join and the meet of the lattice are induced respectively by the following operation on mass problems:

- $A \vee B := \{\langle a, b \rangle : a \in A \text{ and } b \in B\}$ ,
- $A \wedge B := A \sqcup B = \{0 \hat{\ } a : a \in A\} \cup \{1 \hat{\ } b : b \in B\}$ , where  $n \hat{\ } f(0) := n$  and  $n \hat{\ } f(i+1) := f(i)$ .

Next we want to show that the lattice of Medvedev degrees is not a Heyting algebra (see [Sor96, Thm. 9.2]) but only a co-Heyting algebra (i.e. a Brouwer algebra, see [Sor96, Thm. 9.1]), where the subtraction operation is defined as  $A \setminus B := \min\{C : B \leq A \vee B\}$ . In other words, this subtraction is an ‘implication’ with respect to the join of the lattice.

**Proposition 4.4** (Medvedev co-Heyting algebra). *For every set  $X$ ,  $\mathfrak{M}(X)$  is a co-Heyting algebra, where:*

- (1)  $\perp := x \mapsto \mathbb{A}$
- (2)  $\top := x \mapsto \emptyset$ ;
- (3)  $(\varphi \wedge \psi)(x) := \{\langle p_1, a \rangle : a \in \varphi(x)\} \cup \{\langle p_2, b \rangle : b \in \psi(x)\}$ , where  $p_1, p_2$  are two fixed (different) elements in  $\mathbb{A}'$ .
- (4)  $(\varphi \vee \psi)(x) := \{\langle a, b \rangle : a \in \varphi(x) \text{ and } b \in \psi(x)\}$ ;
- (5)  $(\varphi \setminus \psi)(x) := \{c \in \mathbb{A} : (\forall b \in \psi(x))(c \cdot b \in \varphi(x))\}$ .

*Proof.* This proposition can be proved essentially the same way one proves that the Medvedev degrees form a co-Heyting algebra (see [Sor96, Thm. 1.3]). Let  $\varphi, \psi \in \mathfrak{M}(X)$ .

- (1) The reduction  $\perp \leq_{\text{M}} \varphi$  is witnessed by the identity functional.
- (2) The reduction  $\varphi \leq_{\text{M}} \top$  is trivially witnessed by any  $\bar{a} \in \mathbb{A}'$ , as the quantification on  $b \in \top(x)$  is vacuously true.

- (3) The reductions  $\varphi \wedge \psi \leq_M \varphi$  and  $\varphi \wedge \psi \leq_M \psi$  are witnessed respectively by the maps  $a \mapsto \langle p_1, a \rangle$  and  $b \mapsto \langle p_2, b \rangle$ . Moreover, if  $\rho \leq_M \varphi$  via  $a_\varphi$  and  $\rho \leq_M \psi$  via  $a_\psi$  then the reduction  $\rho \leq_M \varphi \wedge \psi$  is witnessed by the map that, upon input  $\langle p, c \rangle$ , if  $p = p_1$  returns  $a_\varphi \cdot c$ , otherwise returns  $a_\psi \cdot c$ .
- (4) The reductions  $\varphi \leq_M \varphi \vee \psi$  and  $\psi \leq_M \varphi \vee \psi$  are witnessed by the projections. Moreover, if  $\varphi \leq_M \rho$  via  $a_\varphi$  and  $\psi \leq_M \rho$  via  $a_\psi$  then  $\varphi \vee \psi \leq_M \rho$  is witnessed by the map  $x \mapsto \langle a_\varphi \cdot x, a_\psi \cdot x \rangle$ .
- (5) We need to show that, for every  $\rho$ ,  $\varphi \setminus \psi \leq_M \rho \iff \varphi \leq_M \psi \vee \rho$ . To prove the left-to-right implication, observe that if  $\bar{a} \in \mathbb{A}'$  witnesses the reduction  $\varphi \setminus \psi \leq_M \rho$ , then, for every  $\langle b, d \rangle \in (\psi \vee \rho)(x)$ ,  $\bar{a} \cdot d \in (\varphi \setminus \psi)(x)$ , and therefore  $(\bar{a} \cdot d) \cdot b \in \varphi(x)$ . To prove the right-to-left implication, notice that if  $\bar{b}$  witnesses  $\varphi \leq_M \psi \vee \rho$ , then, by definition, for every  $\langle b, c \rangle \in (\psi \vee \rho)(x)$ ,  $\bar{b} \cdot \langle b, c \rangle \in \varphi(x)$ . This implies that the map  $b \mapsto \bar{b} \cdot \langle b, c \rangle \in (\varphi \setminus \psi)(x)$ , therefore concluding the proof.  $\square$

Now we want to show some structural properties of Medvedev doctrines. Using the two theorems we recalled from previous work, we can show:

**Proposition 4.5.** *The Medvedev doctrine  $\mathfrak{M}: \text{Set}^{\text{op}} \longrightarrow \text{Pos}$  is a full universal and pure existential doctrine. In particular, for every function  $f: X \rightarrow Y$ , the morphism  $\forall_f: \mathfrak{M}(X) \rightarrow \mathfrak{M}(Y)$  sending an element  $\varphi \in \mathfrak{M}(X)$  to the element  $\forall_f(\varphi) \in \mathfrak{M}(Y)$  defined as*

$$\forall_f(\varphi)(y) := \bigcup_{x \in f^{-1}(y)} \varphi(x)$$

is right adjoint to  $\mathfrak{M}_f$ , i.e.  $\psi \leq_M \forall_f(\varphi) \iff \mathfrak{M}_f(\psi) \leq_M \varphi$  for any  $\psi \in \mathfrak{M}(Y)$  and  $\varphi \in \mathfrak{M}(X)$ .

Similarly, if  $f$  is surjective then the assignment

$$\exists_f(\varphi)(y) := \bigcap_{x \in f^{-1}(y)} \varphi(x)$$

determines a left adjoint to  $\mathfrak{M}_f$ , i.e.  $\exists_f(\varphi) \leq_M \psi \iff \varphi \leq_M \mathfrak{M}_f(\psi)$  for any  $\psi \in \mathfrak{M}(Y)$  and  $\varphi \in \mathfrak{M}(X)$ .

*Proof.* This is essentially a definition-chasing exercise. Let us first show that, for every  $\psi \in \mathfrak{M}(Y)$  and  $\varphi \in \mathfrak{M}(X)$ ,  $\psi \leq_M \forall_f(\varphi) \iff \mathfrak{M}_f(\psi) \leq_M \varphi$ . Assume first that the reduction  $\psi \leq_M \forall_f(\varphi)$  is witnessed by  $\bar{a} \in \mathbb{A}'$ . The same  $\bar{a}$  witnesses  $\mathfrak{M}_f(\psi) \leq_M \varphi$ : indeed, for every  $x \in X$  and every  $a \in \varphi(x)$ , we have  $a \in \forall_f(\varphi)(f(x))$ , and therefore  $\bar{a} \cdot a \in \psi(f(x)) = \mathfrak{M}_f(\psi)(x)$ . On the other hand, assume  $\bar{b}$  witnesses  $\mathfrak{M}_f(\psi) \leq_M \varphi$ . For every  $y \in Y$ , if  $b \in \forall_f(\varphi)(y)$  then  $b \in \varphi(x)$  for some  $x \in f^{-1}(y)$ . In particular,  $\bar{b} \cdot b \in \mathfrak{M}_f(\psi)(x) = \psi(y)$ , i.e.  $\bar{b}$  witnesses  $\psi \leq_M \forall_f(\varphi)$ .

The second part of the statement is proved analogously. Assume first that  $\exists_f(\varphi) \leq_M \psi$  is witnessed by  $\bar{c}$ . Fix  $x \in X$  and  $a \in \mathfrak{M}_f(\psi)(x) = \psi(f(x))$ . In particular,  $\bar{c} \cdot a \in \exists_f(\varphi)(f(x))$ , and hence  $\bar{c} \cdot a \in \varphi(x)$ . Finally, assume  $\bar{d}$  witnesses  $\varphi \leq_M \mathfrak{M}_f(\psi)$ , and let  $y \in Y$  and  $b \in \psi(y)$ . Since  $f$  is surjective, there is  $x \in X$  s.t.  $f(x) = y$ . Moreover, for every  $x \in f^{-1}(y)$  we have  $b \in \psi(f(x))$  and, hence,  $\bar{d} \cdot b \in \varphi(x)$ . This implies that  $\bar{d} \cdot b \in \bigcap_{x \in f^{-1}(y)} \varphi(x) = \exists_f(\varphi)(y)$ .  $\square$

The rest of this section is devoted to studying abstract **universal** properties of Medvedev doctrines.

**Theorem 4.6** (Medvedev isomorphism). *Let  $\mathfrak{M}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  be the Medvedev doctrine for a given PCA  $\mathbb{A}$ . Then we have the isomorphism of full universal doctrines*

$$\mathfrak{M} \equiv (\mathbb{A}_M)^{\forall_f}.$$

In particular, for every set  $X$ , the map

$$(f, \alpha) \mapsto \alpha \circ f^{-1}$$

is a surjective (pre)order homomorphism between  $((\mathbb{A}_M)^{\forall_f}(X), \leq_{\forall_f})$  and  $(\mathfrak{M}(X), \leq_M)$ .

*Proof.* Observe that the full universal completion of  $\mathbb{A}_M$ , i.e. the doctrine  $(\mathbb{A}_M)^{\forall_f}$ , can be described as follows:

- $(\mathbb{A}_M)^{\forall_f}(X) = \{(f, \alpha) : f: Y \rightarrow X \text{ and } \alpha \in \mathbb{A}_M(Y)\}$ ,
- if  $f: Y \rightarrow X$ ,  $\alpha \in \mathbb{A}_M(Y)$ ,  $g: Z \rightarrow X$ , and  $\beta \in \mathbb{A}_M(Z)$ , then

$$(f, \alpha) \leq_{\forall_f} (g, \beta) \iff (\exists h: Z \rightarrow Y)((\forall z \in Z)(g(z) = f \circ h(z)) \text{ and } (\mathbb{A}_M)_h(\alpha) \leq_{\text{dM}} \beta).$$

In other words,  $(f, \alpha) \leq_{\forall_f} (g, \beta)$  if and only if there are  $h: Z \rightarrow Y$  and  $\bar{a} \in \mathbb{A}$  such that the following diagram commutes

$$\begin{array}{ccc} Y & \xleftarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \\ \alpha \downarrow & & \downarrow \beta \\ \mathbb{A} & \xleftarrow{\bar{a}} & \mathbb{A} \end{array}$$

If we define  $\varphi := \alpha \circ f^{-1}$  and  $\psi := \beta \circ g^{-1}$ , we obtain the following diagram:

$$\begin{array}{ccc} Y & \xleftarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \\ \alpha \downarrow & \swarrow \varphi & \searrow \psi \\ \mathbb{A} & \xleftarrow{\bar{a}} & \mathbb{A} \end{array}$$

where  $\varphi$  and  $\psi$  are represented as double arrows to stress the fact that they are maps into  $\wp(\mathbb{A})$ .

Let us first show that  $(f, \alpha) \leq_{\forall_f} (g, \beta)$  implies  $\varphi \leq_M \psi$ . Fix two witnesses  $h: Z \rightarrow Y$  and  $\bar{a} \in \mathbb{A}'$  for  $(f, \alpha) \leq_{\forall_f} (g, \beta)$  and fix  $x \in X$ . If  $x \notin \text{ran}(g)$  then  $\psi(x) = \emptyset$ , hence there is nothing to prove. Assume therefore that  $x \in \text{ran}(g)$ . By definition,  $\psi(x) = \beta(g^{-1}(x)) \neq \emptyset$ . Fix  $b \in \psi(x)$  and let  $z \in g^{-1}(x)$  be s.t.  $\beta(z) = b$ . Since  $(f, \alpha) \leq_{\forall_f} (g, \beta)$ , we can write

$$\bar{a} \cdot b = \bar{a} \cdot \beta(z) = \alpha \circ h(z) = \alpha(y),$$

for some  $y \in f^{-1}(x)$ . In particular,  $\alpha(y) \in \varphi(x)$ , and therefore  $\bar{a}$  witnesses  $\varphi \leq_M \psi$ .

Let us now prove the other direction. Let  $\bar{b} \in \mathbb{A}'$  be a witness for  $\varphi \leq_M \psi$ . Fix  $z \in Z$ . Clearly, letting  $x_z := g(z)$ ,  $\psi(x_z) \neq \emptyset$ . Moreover,  $\psi(x_z) \neq \emptyset$  implies  $\varphi(x_z) = \alpha(f^{-1}(x_z)) \neq \emptyset$  (as  $\bar{b} \cdot \psi(x_z) \subset \varphi(x_z)$ ). We define  $h$  as a choice function that maps  $z$  to some  $y \in f^{-1}(x_z)$  such that  $\alpha(y) \in \bar{b} \cdot \psi(x_z)$ . Observe that  $h$  is well-defined as if  $a \in \bar{b} \cdot \psi(x_z)$  then  $a = \alpha(y)$  for some  $y \in f^{-1}(x_z)$ . This also shows that  $h$  and  $\bar{b}$  witness  $(f, \alpha) \leq_{\forall_f} (g, \beta)$ .

Finally, to show that the homomorphism is surjective it is enough to notice that every  $\varphi: X \rightarrow \wp(\mathbb{A})$  is the image of the pair  $(\pi_X, \pi_{\mathbb{A}})$ , where  $Y := \{(x, a) \in X \times \mathbb{A} : a \in \varphi(x)\}$ , and  $\pi_X: Y \rightarrow X$  and  $\pi_{\mathbb{A}}: Y \rightarrow \mathbb{A}$  are the two projections.  $\square$

In other words, this shows that the Medvedev doctrine can be obtained as the full universal completion of a doctrine.

**Remark 4.7.** Notice that, when we consider a PCA  $\mathbb{A}$  and the trivial elementary sub-PCA given by  $\mathbb{A}$  itself, we have that the Medvedev doctrine  $\mathfrak{M}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  can be presented as the functor obtained by composing the realizability doctrine  $\mathcal{R}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  defined in Example 3.4 with the op-functor  $(-)^{\text{op}}: \mathbf{Pos} \rightarrow \mathbf{Pos}$  inverting the order of posets.

Since it is known that the fibres of the realizability doctrine have the Heyting structure, this presentation provides, for example, an abstract explanation of the co-Heyting structure of the fibres of Medvedev doctrines presented in Proposition 4.4.

**4.1. Muchnik Doctrines.** Using a very similar strategy, we can show that the **Muchnik lattice** is isomorphic to the full universal completion of a doctrine. The notion of *Muchnik reducibility* dates back formally to 1963 but was probably known earlier, and can be thought of as the non-uniform version of Medvedev reducibility. More precisely, given two mass problems  $P, Q \subset \mathbb{N}^{\mathbb{N}}$ , we say that  $P$  is Muchnik reducible to  $Q$ , and write  $P \leq_w Q$ , if for every  $q \in Q$  there is a functional  $\Phi$  such that  $\Phi(q) \in P$ . In other words,  $P \leq_w Q$  if every element of  $Q$  computes some element of  $P$ . Muchnik reducibility is sometimes called “weak reducibility” (which motivates the choice of the symbol  $\leq_w$ ) to contrast with Medvedev reducibility, sometimes called “strong reducibility” [Hin12].

Similarly to the Medvedev degrees, the Muchnik degrees form a distributive lattice, where the join and the meet operations are induced by the same operations on subsets of  $\mathbb{N}^{\mathbb{N}}$  that induce the join and the meet in the Medvedev degrees. Unlike the Medvedev lattice, the Muchnik lattice is both a Heyting and a co-Heyting algebra [Hin12, Prop. 4.3 and 4.7].

**Definition 4.8** (Muchnik doctrine of singletons). Let  $\mathbb{A}$  be a PCA and let  $\mathbb{A}'$  be an elementary sub-PCA. We can define a doctrine  $\mathbb{A}_w: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  mapping a set  $X$  to the set  $\mathbb{A}^X$  of functions from  $X$  to  $\mathbb{A}$ . Given two elements  $\alpha, \beta \in \mathbb{A}_w(X)$ , we define  $\alpha \leq_{\text{dw}} \beta$  if

$$(\forall x \in X)(\exists \bar{\alpha} \in \mathbb{A}')(\bar{\alpha} \cdot \beta(x) = \alpha(x)).$$

The functor  $\mathbb{A}_w: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  is called **Muchnik doctrine of singletons**.

**Definition 4.9** (Muchnik doctrine). We define the **Muchnik doctrine**  $\mathfrak{M}_w: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  over  $\mathbf{Set}$  as follows. For every set  $X$  and every pair of functions  $\varphi, \psi \in \wp(\mathbb{A})^X$ , we define

$$\varphi \leq_w \psi : \iff (\forall x \in X)(\forall b \in \psi(x))(\exists \bar{\alpha} \in \mathbb{A}')(\bar{\alpha} \cdot b \in \varphi(x)).$$

This preorder induces an equivalence relation on functions in  $\wp(\mathbb{A})^X$ . The doctrine  $\mathfrak{M}_w(X)$  is defined as the quotient of  $\wp(\mathbb{A})^X$  by the equivalence relation generated by  $\leq_w$ . The partial order on the equivalence classes  $[\varphi]$  is the one induced by  $\leq_w$ .

**Proposition 4.10.** *For every set  $X$ ,  $\mathfrak{M}_w(X)$  is both a Heyting and a co-Heyting algebra, where:*

- (1)  $\perp := x \mapsto \mathbb{A}$
- (2)  $\top := x \mapsto \emptyset$ ;

- (3)  $(\varphi \wedge \psi)(x) := \{\langle p_1, a \rangle : a \in \varphi(x)\} \cup \{\langle p_2, b \rangle : b \in \psi(x)\}$ , where  $p_1, p_2$  are two fixed (different) elements in  $\mathbb{A}'$ .
- (4)  $(\varphi \vee \psi)(x) := \{\langle a, b \rangle : a \in \varphi(x) \text{ and } b \in \psi(x)\}$ ;
- (5)  $(\varphi \setminus \psi)(x) := \{c \in \mathbb{A} : (\forall b \in \psi(x))(c \cdot b \in \varphi(x))\}$ .
- (6)  $(\varphi \rightarrow \psi)(x) := \{b \in \psi(x) : (\forall \bar{a} \in \mathbb{A}')(\bar{a} \cdot b \notin \varphi(x))\}$ .

*Proof.* The points 1-5 can be proved as in the proof of Proposition 4.4, so we only prove point 6. The argument is a straightforward generalization of [Hin12, Prop. 4.3]. We need to show that, for every  $\rho$ ,  $\rho \leq_w \varphi \rightarrow \psi \iff \varphi \wedge \rho \leq_w \psi$ . Observe that,

$$\begin{aligned}
\rho \leq_w \varphi \rightarrow \psi &\iff (\forall x \in X)(\forall b \in (\varphi \rightarrow \psi)(x))(\exists \bar{a} \in \mathbb{A}')(\bar{a} \cdot b \in \rho(x)) \\
&\iff (\forall x \in X)(\forall b \in \psi(x))((\forall \bar{a} \in \mathbb{A}')(\bar{a} \cdot b \notin \varphi(x)) \Rightarrow (\exists \bar{b} \in \mathbb{A}')(\bar{b} \cdot b \in \rho(x))) \\
&\iff (\forall x \in X)(\forall b \in \psi(x))((\exists \bar{a} \in \mathbb{A}')(\bar{a} \cdot b \in \varphi(x)) \vee (\exists \bar{b} \in \mathbb{A}')(\bar{b} \cdot b \in \rho(x))) \\
&\iff (\forall x \in X)(\forall b \in \psi(x))(\exists \bar{c} \in \mathbb{A}')((\exists a \in \varphi(x))(\bar{c} \cdot b = \langle p_1, a \rangle) \vee (\exists d \in \rho(x))(\bar{c} \cdot b = \langle p_2, d \rangle)) \\
&\iff (\forall x \in X)(\forall b \in \psi(x))(\exists \bar{c} \in \mathbb{A}')(\bar{c} \cdot b \in \varphi \wedge \rho(x)) \\
&\iff \varphi \wedge \rho \leq_w \psi. \quad \square
\end{aligned}$$

It is straightforward to adapt the proof of Theorem 4.6 to show the following:

**Theorem 4.11** (Muchnik isomorphism). *The Muchnik doctrine  $\mathfrak{M}_w$  is isomorphic to the full universal completion  $(\mathbb{A}_w)^{\forall_f}$ .*

Observe that the Medvedev doctrine of singletons  $\mathbb{A}_M : \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  in Definition 4.2 can be embedded in the Muchnik doctrine of singletons  $\mathbb{A}_w : \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  as in Definition 4.8. Let us denote such a morphism of doctrines by  $\mathfrak{J} : \mathbb{A}_M \rightarrow \mathbb{A}_w$ . By Theorem 3.7 we can employ the universal property of the full universal completion. Thus we conclude the following corollary.

**Corollary 4.12.** *Let  $\mathbb{A}$  be a given PCA and let  $\mathbb{A}'$  be an elementary sub-PCA. Then there exists a morphism of full universal doctrines such that the diagram*

$$\begin{array}{ccc}
& & \mathfrak{M} \\
& \nearrow \eta_{\mathbb{A}_M}^{\forall_f} & \downarrow \\
\mathbb{A}_M & \xrightarrow{\mathfrak{J}} & \mathbb{A}_w \xrightarrow{\eta_{\mathbb{A}_w}^{\forall_f}} \mathfrak{M}_w \\
& & \downarrow \Upsilon
\end{array}$$

*commutes.*

This corollary is the categorical version of how Medvedev (strong) reducibility is the uniform version of Muchnik (weak) reducibility.

## 5. WEIHRAUCH DOCTRINES

Weihrauch degrees are important because multi-valued functions on represented spaces can be considered as realizers of mathematical theorems in a very natural way, and studying the Weihrauch reductions between theorems is like asking which theorems can be transformed continuously or computably into another, as explained in [BG11]. This provides a purely topological or computational approach to metamathematics that sheds new light on the

nature of theorems. Hence it is natural to ask whether, with a similar approach as the one from the last section, we can also describe *Weihrauch reducibility* as the completion of some existential doctrine.

Weihrauch reducibility is a notion of reducibility on multi-valued functions on represented spaces. This reducibility is useful to study the uniform computational strength of problems. While Weihrauch reducibility is often introduced in the context of Type-2 computability, we will introduce it in the more general context of PCAs, as we did for the Medvedev reducibility relation.

If  $X, Y, Z, W$  are in  $\text{Mod}(\mathbb{A}, \mathbb{A}')$  and  $f : \subseteq X \rightrightarrows Y$ ,  $g : \subseteq Z \rightrightarrows W$  are partial multi-valued functions, we say that  $f$  is **Weihrauch reducible** to  $g$ , and write  $f \leq_W g$ , if there are two morphisms of modest sets (i.e. two computable functionals)  $\Phi, \Psi \in \mathbb{A}'$  s.t.

$$(\forall p \in \text{dom}(f \circ \delta_X))(\forall G \vdash g)((p \mapsto \Psi(p, G\Phi(p))) \vdash f),$$

where  $G \vdash g$  means that  $G$  is a realizer of  $g$  (as defined in 2.8).

We can rephrase the definition of Weihrauch reducibility without explicitly<sup>2</sup> mentioning the realizers as follows:

$$(\forall p \in \text{dom}(f \circ \delta_X))(\Phi(p) \in \text{dom}(g \circ \delta_Z) \wedge (\forall q \in g \circ \delta_Z(\Phi(p)))(\Psi(p, q) \in f \circ \delta_X(p))).$$

**Remark 5.1.** Observe that, if we are only interested in studying the degree structure induced by Weihrauch reducibility, we can restrict our attention to functions of the type  $f: X \rightrightarrows \mathbb{A}$ , i.e.  $f: X \rightarrow \wp(\mathbb{A})$  such that  $f(x) \neq \emptyset$  for every  $X$ . Indeed,  $f : \subseteq X \rightrightarrows Y$  is Weihrauch-equivalent to  $f' : \text{dom}(f) \rightrightarrows \mathbb{A}$  defined as  $f'(x) := \delta_Y^{-1}(f(x)) = \{a \in \mathbb{A} : \delta_Y(a) \in f(x)\}$ .

Notice that, by restricting our attention to total multi-valued functions, if  $f \leq_W g$  is witnessed by the functionals  $\Phi, \Psi$  as above, we can always assume that  $\delta_X^{-1}(X) \subset \text{dom}(\Phi)$ . For our purposes, it is convenient to rewrite the definition of Weihrauch reducibility in the following (equivalent) form.

**Definition 5.2** (Weihrauch reducibility relation). Let  $(X, \delta_X)$  and  $(Z, \delta_Z)$  be in  $\text{Mod}(\mathbb{A}, \mathbb{A}')$ . Given two multi-valued functions  $f: X \rightarrow \wp^*(\mathbb{A})$  and  $g: Z \rightarrow \wp^*(\mathbb{A})$  then  $f \leq_W g$  if and only if there is a morphism  $k: X \rightarrow Z$  of modest sets and an element  $\bar{a} \in \mathbb{A}'$  s.t.

$$(\forall p \in \text{dom}(\delta_X))(\forall q \in g \circ k \circ \delta_X(p))(\bar{a} \cdot \langle p, q \rangle \in f \circ \delta_X(p)).$$

We want to generalize this definition similarly to what we did for the Medvedev reducibility. To this end, we introduce the following doctrine.

**Definition 5.3** (elementary Weihrauch doctrine). Let  $\mathbb{A}$  be a PCA and  $\mathbb{A}' \subset \mathbb{A}$  be an elementary sub-PCA of  $\mathbb{A}$ . We define the **elementary Weihrauch doctrine**  $\mathfrak{e}\mathfrak{W}: \text{Mod}(\mathbb{A}, \mathbb{A}')^{\text{op}} \longrightarrow \text{Pos}$  as follows: for every modest set  $(X, \delta_X)$  and every pair of functions  $f, g$  in  $\wp^*(\mathbb{A})^X$ , we define

$$f \leq_{\text{dW}} g : \iff (\exists \bar{a} \in \mathbb{A}')(\forall p \in \text{dom}(\delta_X))(\forall q \in g \circ \delta_X(p))(\bar{a} \cdot \langle p, q \rangle \in f \circ \delta_X(p)).$$

This preorder induces an equivalence relation on functions in  $\wp^*(\mathbb{A})^X$ . The doctrine  $\mathfrak{e}\mathfrak{W}(X, \delta_X)$  is defined as the quotient of  $\wp^*(\mathbb{A})^X$  by the equivalence relation generated by  $\leq_{\text{dW}}$ . The partial order on the equivalence class  $[f]$  is the one induced by  $\leq_{\text{dW}}$ .

<sup>2</sup>The existence of a realizer for every multi-valued function depends on a (relatively) weak form of the axiom of choice.

Notice that following the same ideas used to prove Proposition 4.5, we can prove the following proposition:

**Proposition 5.4.** *The elementary Weihrauch doctrine  $\mathfrak{e}\mathfrak{W}: \text{Mod}(\mathbb{A}, \mathbb{A}')^{\text{op}} \longrightarrow \text{Pos}$  is a pure universal and pure existential doctrine. In particular, for every projection  $\pi_Y: X \times Y \rightarrow Y$  of modest sets, the morphism  $\forall_{\pi_Y}: \mathfrak{e}\mathfrak{W}(X \times Y) \rightarrow \mathfrak{e}\mathfrak{W}(Y)$  sending an element  $f \in \mathfrak{e}\mathfrak{W}(X \times Y)$  to the element  $\forall_{\pi_Y}(f) \in \mathfrak{e}\mathfrak{W}(Y)$  defined as*

$$\forall_{\pi_Y}(f)(y) := \bigcup_{x \in X} f(x, y)$$

*is right adjoint to  $\mathfrak{e}\mathfrak{W}_{\pi_Y}$ , i.e.  $g \leq_{\text{dW}} \forall_{\pi_Y}(f) \iff \mathfrak{e}\mathfrak{W}_{\pi_Y}(g) \leq_{\text{dW}} f$  for any  $g \in \mathfrak{e}\mathfrak{W}(Y)$  and  $f \in \mathfrak{e}\mathfrak{W}(X \times Y)$ .*

*Similarly, the assignment*

$$\exists_{\pi_Y}(f)(y) := \bigcap_{x \in X} f(x, y)$$

*determines a left adjoint to  $\mathfrak{e}\mathfrak{W}_{\pi_Y}$ , i.e.  $\exists_{\pi_Y}(f) \leq_{\text{dW}} g \iff f \leq_{\text{dW}} \mathfrak{e}\mathfrak{W}_{\pi_Y}(g)$  for any  $g \in \mathfrak{e}\mathfrak{W}(Y)$  and  $f \in \mathfrak{e}\mathfrak{W}(X \times Y)$ .*

Now we introduce the Weihrauch doctrine. For this, we introduce first the notion of *generalized Weihrauch predicate* on a modest set.

**Definition 5.5** (Weihrauch predicate). Let  $(X, \delta_X)$  be a modest set. A **generalized Weihrauch predicate** on  $(X, \delta_X)$  is a function

$$F: X \rightarrow \wp^*(\mathbb{A})^Y$$

for some modest set  $(Y, \delta_Y)$ .

Given a generalized Weihrauch predicate  $F$ , we denote by  $F_x: Y \rightarrow \wp^*(\mathbb{A})$  the function  $F(x)$ . Moreover, with a small abuse of notation we write  $F(x, y)$  for  $F(x)(y)$ .

**Definition 5.6** (Weihrauch doctrine). Given a PCA  $\mathbb{A}$  with elementary sub-PCA  $\mathbb{A}'$ , the **Weihrauch doctrine** is the functor  $\mathfrak{W}: \text{Mod}(\mathbb{A}, \mathbb{A}')^{\text{op}} \longrightarrow \text{Pos}$  that maps a modest set  $(X, \delta_X)$  to the preorder  $\mathfrak{W}(X, \delta_X)$  defined as follows:

- objects are generalized Weihrauch predicates on  $(X, \delta_X)$ ;
- the partial order is given by the poset reflection of the preorder defined as follows: let  $(Y, \delta_Y)$  and  $(Z, \delta_Z)$  be modest sets. For every  $F: X \rightarrow \wp^*(\mathbb{A})^Y$  and  $G: X \rightarrow \wp^*(\mathbb{A})^Z$ , we say that  $F \leq_{\text{W}} G$  if there exist a morphism of modest sets  $k: X \times Y \rightarrow Z$  and  $\bar{a} \in \mathbb{A}'$  such that

$$(\forall p \in \text{dom}(\delta_{X \times Y}))(\forall q \in G \circ \langle \pi_X, k \rangle \circ \delta_{X \times Y}(p))(\bar{a} \cdot \langle p, q \rangle \in F \circ \delta_{X \times Y}(p)).$$

In other words, if  $F$  and  $G$  are generalized Weihrauch predicates, the reduction  $F \leq_{\text{W}} G$  can be seen as a uniform sequence of Weihrauch reductions, one for each  $x \in X$ , all witnessed by the same reduction functionals. This observation makes the following result apparent.

**Theorem 5.7** (Weihrauch lattice). *Let  $(\mathbb{A}, \mathbb{A}') = (\mathbb{N}^{\mathbb{N}}, \mathbb{N}_{\text{eff}}^{\mathbb{N}})$  be Kleene's second model. The Weihrauch lattice is isomorphic to  $\mathfrak{W}(1)$ .*

*Proof.* By definition of Weihrauch doctrine, the objects of the poset  $\mathfrak{W}(1)$  can be identified with functions  $Y \rightarrow \wp^*(\mathbb{A})$ . Moreover, if  $F: Y \rightarrow \wp^*(\mathbb{A})$  and  $G: Z \rightarrow \wp^*(\mathbb{A})$  are in  $\mathfrak{W}(1)$

then  $F \leq_W G$  iff there are a morphism  $k: (Y, \delta_Y) \rightarrow (Z, \delta_Z)$  of modest sets and an effective map  $\bar{a} \in \mathbb{A}'$  such that

$$(\forall p \in \text{dom}(\delta_Y))(\forall q \in G \circ k \circ \delta_Y(p))(\bar{a} \cdot \langle p, q \rangle \in F \circ \delta_X(p)),$$

which corresponds to Definition 5.2.  $\square$

Notice that it is direct to check that Weihrauch doctrines are pure existential doctrines with the following assignment:

**Proposition 5.8.** *The Weihrauch doctrine  $\mathfrak{W}: \text{Mod}(\mathbb{A}, \mathbb{A}')^{\text{op}} \longrightarrow \text{Pos}$  is pure existential. In particular, for every projection  $\pi_Y: X \times Y \rightarrow Y$  and every generalized Weihrauch predicate  $F: X \times Y \rightarrow \wp(\mathbb{A})^Z$ , the predicate  $\exists_{\pi_Y}(F)$  is defined as the function*

$$\exists_{\pi_Y}(F): Y \rightarrow \wp^*(\mathbb{A})^{(X \times Z)}$$

sending an element  $y \in Y$  into the function  $\exists_{\pi_Y}(F)(y): X \times Z \rightarrow \wp^*(\mathbb{A})$  sending  $(x, z) \mapsto F(x, y)(z)$ .

Now we show that the Weihrauch doctrine is isomorphic to the pure existential completion of the elementary Weihrauch doctrine in Definition 5.3. Recall that, by definition, the elements of  $\mathfrak{eW}^{\exists}(X, \delta_X)$  are pairs  $(\pi, f)$  where  $\pi: (X, \delta_X) \times (Y, \delta_Y) \rightarrow (X, \delta_X)$  is a projection on  $(X, \delta_X)$  in the category  $\text{Mod}(\mathbb{A}, \mathbb{A}')$  and  $f \in \mathfrak{eW}(X \times Y, \delta_{X \times Y})$ , i.e.  $f: X \times Y \rightarrow \wp^*(A)$ . Moreover,

$$(\pi, f) \leq_{\exists} (\pi', g) \iff (\exists k: X \times Y \rightarrow Z)((\forall (x, y) \in X \times Y)(\exists z \in Z) \text{ and } f \leq_{\text{dW}} \mathfrak{eW}_{\langle \pi_X, k \rangle}(g))$$

**Theorem 5.9** (Weihrauch isomorphism). *Let  $\mathbb{A}$  be a PCA and let  $\mathbb{A}'$  be an elementary sub-PCA of  $\mathbb{A}$ . The Weihrauch doctrine is isomorphic to the pure existential completion of the elementary Weihrauch doctrine. In symbols:*

$$\mathfrak{W} \equiv \mathfrak{eW}^{\exists}.$$

In particular, for every modest set  $(X, \delta_X)$ , the map

$$(\pi_X, f) \mapsto F$$

where  $F: X \rightarrow \wp(\mathbb{A})^Y$  is defined as  $F(x) := f(x, \cdot)$ , is a surjective (pre)order homomorphism between  $(\mathfrak{eW}^{\exists}(X), \leq_{\exists})$  and  $(\mathfrak{W}(X), \leq_W)$ .

*Proof.* We first show that the assignment  $(\pi_X, f) \mapsto (y \mapsto f(x, y))$  preserves the order. Let  $(\pi, f)$  and  $(\pi', g)$  be two elements of  $\mathfrak{eW}^{\exists}(X)$ . By definition,  $(\pi, f) \leq_{\exists} (\pi', g)$  if and only if  $(\exists k: X \times Y \rightarrow Z)((\forall (x, y) \in X \times Y)(\exists z \in Z)$  such that  $f \leq_{\text{dW}} \mathfrak{eW}_{\langle \pi_X, k \rangle}(g)$ , i.e.

$$(\exists \bar{a} \in \mathbb{A}')(\forall p \in \text{dom}(\delta_{X \times Y}))((\forall q \in g \circ \langle \pi_X, k \rangle \circ \delta_{X \times Y}(p))(\bar{a} \cdot \langle p, q \rangle \in f \circ \delta_{X \times Y}(p)). \quad (5.1)$$

Let  $F: X \rightarrow \wp^*(\mathbb{A})^Y$  and  $G: X \rightarrow \wp^*(\mathbb{A})^Z$  be the images of  $(\pi, f)$  and  $(\pi', g)$  respectively. In particular,  $F(x) := f(x, \cdot)$  and  $G(x) := g(x, \cdot)$ . Hence, by substituting  $F$  and  $G$  in the equation (5.1), it is straightforward to check that  $(\pi, f) \leq_{\exists} (\pi', g)$  iff  $F \leq_W G$ .

This shows that the embedding preserves and reflects the partial order. Finally, it is easy to see that the map  $(\pi_X, f) \mapsto F$  is surjective, since any function  $H: X \rightarrow \wp^*(\mathbb{A})^V$  can be obtained via our embedding as the image of the pair  $(\pi_X, h)$ , where  $h: X \times V \rightarrow \wp^*(\mathbb{A})$  is defined as  $h(x, v) := H(x, v)$ .  $\square$

Recall by [Hyl88] that the category of modest sets is cartesian closed. Moreover, recall that the pure existential completion preserves the pure universal structure when the starting doctrine is universal and the base is cartesian closed (see for example [TSdP21] or [Hof11]). Therefore, combining these two facts with Theorem 5.9 and Proposition 5.4 we have the following corollary:

**Corollary 5.10.** *Weihrauch doctrines are pure universal and pure existential doctrines.*

**Remark 5.11.** Notice that the Weihrauch doctrine presented in Definition 5.6 can be considered in a more general context by replacing the category of modest sets with different categories, such as the category of assemblies, for example. In particular, by considering other base categories with suitable morphisms, e.g. continuous functions, we could employ the same tools used to prove the Weihrauch isomorphism theorem to present, for example, the continuous version of Weihrauch reducibility (obtained by only requiring that the two functionals  $\Phi$  and  $\Psi$  are continuous, i.e. computable relatively to some oracle) as the pure existential completion of a doctrine.

**5.1. Strong Weihrauch Doctrines.** In this subsection we want to study a variant of the notion of Weihrauch reducibility, that is, "strong" Weihrauch reducibility. The procedure is similar to the one for Medvedev and Weihrauch reducibilities.

We say that  $f$  is **strongly Weihrauch reducible** to  $g$ , and write  $f \leq_{\text{sW}} g$ , if  $\Psi$  does not have direct access to  $p$ . In symbols,  $f \leq_{\text{sW}} g$  if there are  $\Phi, \Psi \in \mathbb{A}'$  such that

$$(\forall p \in \text{dom}(f \circ \delta_X))(\forall G \vdash g)((p \mapsto \Psi G \Phi(p)) \vdash f).$$

Then we observe that the results of the subsection above can be adapted to show that the **strong Weihrauch degrees** are isomorphic to the pure existential completion of a doctrine. To this end, we adapt the definition of the Weihrauch doctrine as follows:

**Definition 5.12.** Given a PCA  $\mathbb{A}$  with elementary sub-PCA  $\mathbb{A}'$ , we define the **elementary strong Weihrauch doctrine**  $\mathfrak{sW}: \text{Mod}(\mathbb{A}, \mathbb{A}')^{\text{op}} \longrightarrow \text{Pos}$  as follows. For every modest set  $(X, \delta_X)$  and every pair of functions  $f, g \in \wp^*(\mathbb{A})^X$ , we define

$$f \leq_{\text{dsW}} g : \iff (\exists \bar{a} \in \mathbb{A}')(\forall p \in \text{dom}(\delta_X))(\forall q \in g \circ \delta_X(p))(\bar{a} \cdot q \in f \circ \delta_X(p)),$$

This preorder induces an equivalence relation on functions in  $\wp^*(\mathbb{A})^X$ . The doctrine  $\mathfrak{sW}(X)$  is defined as the quotient of  $\wp^*(\mathbb{A})^X$  by the equivalence relation generated by  $\leq_{\text{dsW}}$ . The partial order on the equivalence classes  $[f]$  is the one induced by  $\leq_{\text{dsW}}$ .

**Definition 5.13** (Strong Weihrauch doctrine). Given a PCA  $\mathbb{A}$  with elementary sub-PCA  $\mathbb{A}'$ , the **strong Weihrauch doctrine** is the functor  $\mathfrak{SW}: \text{Mod}(\mathbb{A}, \mathbb{A}')^{\text{op}} \longrightarrow \text{Pos}$  that maps a modest set  $(X, \delta_X)$  to the preorder  $\mathfrak{SW}(X, \delta_X)$  defined as follows:

- objects are generalized Weihrauch predicates on  $(X, \delta_X)$ ;
- the partial order is given by the poset reflection of the preorder defined as follows: let  $(Y, \delta_Y)$  and  $(Z, \delta_Z)$  be modest sets. We say that  $F \leq_{\text{SW}} G$ , where  $F: X \rightarrow \wp^*(\mathbb{A})^Y$  and  $G: X \rightarrow \wp^*(\mathbb{A})^Z$ , if there exists a morphism of modest sets  $k: X \times Y \rightarrow Z$  and  $\bar{a} \in \mathbb{A}'$  such that

$$(\forall p \in \text{dom}(\delta_{X \times Y}))(\forall q \in G \circ \langle \pi_X, k \rangle \circ \delta_{X \times Y}(p))(\bar{a} \cdot q \in F \circ \delta_{X \times Y}(p)).$$

In light of Theorem 5.7 and Theorem 5.9, the following results are straightforward:

**Theorem 5.14.** *Let  $(\mathbb{A}, \mathbb{A}') = (\mathbb{N}^{\mathbb{N}}, \mathbb{N}_{\text{eff}}^{\mathbb{N}})$  be Kleene's second model. The strong Weihrauch lattice is isomorphic to  $\mathfrak{SW}(1)$ .*

**Theorem 5.15** (Strong Weihrauch isomorphism). *Let  $\mathbb{A}$  be a PCA and let  $\mathbb{A}'$  be an elementary sub-PCA of  $\mathbb{A}$ . The strong Weihrauch doctrine is isomorphic to the pure existential completion of the elementary strong Weihrauch doctrine. In symbols:*

$$\mathfrak{SW} \equiv \mathfrak{sW}^{\exists}.$$

**5.2. Extended Weihrauch Reducibility.** Now we would like to relate the work of Bauer in [Bau21] to our notions of doctrines introduced before. Bauer in [Bau21, Def. 3.7] extends the preorder given by Weihrauch reducibility to a preorder on functions  $\mathbb{A} \rightarrow \wp\wp(\mathbb{A})$ , called *extended Weihrauch predicates*. In particular, if  $f, g$  are extended Weihrauch predicates, we say that  $f$  is *extended-Weihrauch reducible* to  $g$ , and write  $f \leq_{\text{eW}} g$  if there are  $\ell_1, \ell_2 \in \mathbb{A}'$  such that

- for every  $r \in \mathbb{A}$  s.t.  $f(r) \neq \emptyset$ ,  $\ell_1 \cdot r \downarrow$  and  $g(\ell_1 \cdot r) \neq \emptyset$ ;
- for every  $\theta \in f(r)$  there is  $\xi \in g(\ell_1 \cdot r)$  such that for every  $s \in \xi$ ,  $\ell_2 \cdot r \cdot s \downarrow$  and  $\ell_2 \cdot r \cdot s \in \theta$ .

Bauer also showed ([Bau21, Prop. 3.9]), that the (classical) Weihrauch degrees embed properly on the extended-Weihrauch degrees. The definition of extended Weihrauch reducibility can be naturally strengthened by requiring that the map  $\ell_2$  does not have access to the original input.

**Definition 5.16** (extended-strong-Weihrauch reducibility). *If  $f, g$  are extended Weihrauch predicates, we say that  $f$  is *extended-strong-Weihrauch reducible* to  $g$ , and write  $f \leq_{\text{esW}} g$  if there are  $\ell_1, \ell_2 \in \mathbb{A}'$  s.t.*

- for every  $r \in \mathbb{A}$  s.t.  $f(r) \neq \emptyset$ ,  $\ell_1 \cdot r \downarrow$  and  $g(\ell_1 \cdot r) \neq \emptyset$ ;
- for every  $\theta \in f(r)$  there is  $\xi \in g(\ell_1 \cdot r)$  s.t. for every  $s \in \xi$ ,  $\ell_2 \cdot s \downarrow$  and  $\ell_2 \cdot s \in \theta$ .

Since the map  $\ell_2$  does not have access to the original input, the definition of extended-strong-Weihrauch degrees could be given in the more general context of functions  $X \rightarrow \wp\wp(\mathbb{A})$ .

Observe that the preorder defined by the extended strong Weihrauch reducibility arises naturally as the full existential completion of the Medvedev doctrine  $\mathfrak{M}$ . To show that, we first prove that the full existential completion of the  $\mathfrak{M}$  is equivalent to a new doctrine  $D$ , and then we show how to write  $\leq_{\text{esW}}$  in terms of  $\leq_D$ .

We define the new doctrine  $D$  as  $D(X) := \wp\wp(\mathbb{A})^X$  and

$$\begin{aligned} H \leq_D K &: \iff (\exists \bar{a} \in \mathbb{A})(\forall x \in X)(\forall U \in H(x))(\exists V \in K(x))(\bar{a} \cdot V \subset U) \\ &\iff (\exists \bar{a} \in \mathbb{A})(\exists \lambda : \subseteq X \times \wp(\mathbb{A}) \rightarrow \wp(\mathbb{A}))(\forall x \in X)(\forall U \in H(x))(\bar{a} \cdot \lambda(x, U) \subset U), \end{aligned}$$

where  $\lambda$  is a choice function that maps every pair  $(x, U)$  with  $x \in X$  and  $U \in H(x)$  to some  $V \in K(x)$ .

**Lemma 5.17.** *For every set  $X$ , the map*

$$(f, \varphi) \mapsto \varphi \circ f^{-1}$$

*is a surjective (pre)order homomorphism between  $(\mathfrak{M}^{\exists}(X), \leq_{\exists_f})$  and the preorder  $(D(X), \leq_D)$  defined above*

*Proof.* Fix  $(f, \varphi), (g, \psi) \in \mathfrak{M}^{\exists_f}(X)$ , with  $\varphi \in \wp(\mathbb{A})^Y$  and  $\psi \in \wp(\mathbb{A})^Z$ , and define  $H := \varphi \circ f^{-1}$  and  $K := \psi \circ g^{-1}$ . We obtain the following diagram:

$$\begin{array}{ccc}
 Y & \xrightarrow{k} & Z \\
 \varphi \downarrow & \begin{array}{c} \searrow f \\ \swarrow g \end{array} & \\
 & X & \\
 & \begin{array}{c} \swarrow H \\ \searrow K \end{array} & \\
 \mathbb{A} & \xleftarrow{\bar{a}} & \mathbb{A}
 \end{array}$$

where double and triple arrows represent, respectively, functions into  $\wp(\mathbb{A})$  or into  $\wp\wp(\mathbb{A})$ .

Let us first show that  $(f, \varphi) \leq_{\exists_f} (g, \psi)$  implies  $H \leq_D K$ . Fix  $k: Y \rightarrow Z$  and  $\bar{a} \in \mathbb{A}$  witnessing  $(f, \varphi) \leq_{\exists_f} (g, \psi)$ . Let  $c: \subseteq X \times \wp(\mathbb{A}) \rightarrow Y$  be a choice function that maps every pair  $(x, U)$  with  $x \in X$  and  $U$  in  $H(x)$ , to some  $y \in \varphi^{-1}(U)$ . By definition, if  $U \in H(x)$  then  $\varphi^{-1}(U) \neq \emptyset$ , hence  $c$  is well-defined. Observe also that if  $y$  is in  $\varphi^{-1}(U)$  then, by definition of  $H$ ,  $f(y) = x$ . We define  $\lambda(x, U) := \psi \circ k \circ c(x, U)$ . We claim that  $\bar{a}$  and  $\lambda$  witness  $H \leq_D K$ . Indeed, for every  $x$  in  $X$  and every  $U$  in  $H(x)$ , letting  $y := c(x, U)$  we have

$$\bar{a} \cdot \lambda(x, U) = \bar{a} \cdot (\psi \circ k(y)) \subset \varphi(y) = U.$$

On the other hand, if  $H \leq_D K$  as witnessed by  $\bar{a}, \lambda$ , then we define  $k: Y \rightarrow Z$  as a choice function that maps every  $y$  in  $Y$  to some element in  $\{z \in Z : \psi(z) = \lambda(f(y), \varphi(y))\}$ . Notice that  $k$  is well-defined: indeed, by hypothesis,  $\lambda(f(y), \varphi(y)) = V \in K(f(y))$ , where  $V = \psi(z)$  for some  $z$  such that  $g(z) = f(y)$ . This also shows that, for every  $y \in Y$ ,  $f(y) = g(k(y))$ . To prove that  $\bar{a}$  and  $k$  witness  $(f, \varphi) \leq_{\exists_f} (g, \psi)$  it is enough to notice that, for every  $y$  in  $Y$ ,  $\bar{a} \cdot \psi(k(y)) = \bar{a} \cdot \lambda(f(y), \varphi(y)) \subset \varphi(y)$ .  $\square$

**Theorem 5.18.** *For every  $H, K \in D(\mathbb{A})$ ,*

$$H \leq_{\text{esW}} K \iff (\exists \bar{k} \in \mathbb{A}')(H \leq_D K \circ \bar{k}).$$

*Proof.* This is immediate by unfolding the definitions. Indeed, if  $H \leq_{\text{esW}} K$  via  $\ell_1, \ell_2$ , then  $\ell_2$  witnesses the reduction  $H \leq_D K \circ \ell_1$ . On the other hand, if  $H \leq_D K \circ \Phi$  via  $\bar{a}$  then the maps  $\ell_1 := \Phi$  and  $\ell_2 := \bar{a}$  witness the reduction  $H \leq_{\text{esW}} K$ .  $\square$

## 6. DIALECTICA DOCTRINES

Finally, we ready to formally establish a first link between the Dialectica doctrines and the doctrines for computability presented in the previous sections, employing categorical universal properties.

Dialectica categories were originally introduced in [dP89] as a categorification of Gödel's Dialectica interpretation [GF<sup>+</sup>86]. Over the years, several authors have noticed some resemblance between the structure of Dialectica categories and some known notions of computability. However, despite the outwardly similar appearance, a formal connection between computability and Dialectica categories has never been proved so far.

Recall from [TSdP22] the following definition of the Dialectica doctrine. This notion is the proof-irrelevant version of a more general construction in the fibrational setting [Hof11, TSdP21].

**Dialectica construction.** Let  $P: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  be a doctrine whose base category  $\mathcal{C}$  has finite products. The **dialectica doctrine**  $\mathfrak{Dial}(P): \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  is defined as the functor sending:

- an object  $I$  into the poset  $\mathfrak{Dial}(P)(I)$  defined as follows:
  - **objects** are quadruples  $(I, U, X, \alpha)$  where  $I, U$  and  $X$  are objects of the base category  $\mathcal{C}$  and  $\alpha \in P(I \times U \times X)$ ;
  - **partial order:** we stipulate that  $(I, U, X, \alpha) \leq (I, V, Y, \beta)$  if there exists a pair  $(f_0, f_1)$ , where  $f_0: I \times U \rightarrow V$  and  $f_1: I \times U \times Y \rightarrow X$  are morphisms of  $\mathcal{C}$  such that:

$$\alpha(i, u, f_1(i, u, y)) \leq \beta(i, f_0(i, u), y).$$

- an arrow  $g: J \rightarrow I$  into the poset morphism  $\mathfrak{Dial}(P)(I) \rightarrow \mathfrak{Dial}(P)(J)$  sending a predicate  $(I, U, X, \alpha)$  to the predicate:

$$(J, U, X, \alpha(g(j), u, x)).$$

In order to understand the intuition behind the notion of Dialectica doctrine, let us consider the poset  $\mathfrak{Dial}(P)(1)$ : an object of this poset  $(1, U, X, \alpha)$  represents a relation  $U \xleftarrow{\alpha} X$  and we have that  $U \xleftarrow{\alpha} X$  is less or equal to  $V \xleftarrow{\beta} Y$  when there exists a *witness* function  $f_0: U \rightarrow V$  and a *counterexample* function  $f_1: U \times Y \rightarrow X$ , graphically

$$\begin{array}{ccc} U & \xleftarrow{\alpha} & X \\ & \searrow & \nearrow f_1 \\ & & Y \\ & \swarrow & \longleftarrow \beta \\ V & & \end{array}$$

$f_0$

such that  $\alpha(u, f_1(u, y)) \subseteq \beta(f_0(u), y)$ .

Now we recall a useful result due to Hofstra [Hof11], who revisited the Dialectica construction and showed that it can be presented as a free construction involving quantifier completions. The original result was presented in the language of fibrations<sup>3</sup>, but here we follow the presentation in [TSdP22] where this theorem is stated in terms of doctrines.

**Theorem 6.1** (Hofstra [Hof11]). *Let  $P: \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$  be any doctrine. There is an isomorphism*

$$\mathfrak{Dial}(P) \cong (P^{\forall})^{\exists}$$

*relating the Dialectica construction over  $P$  to the pure existential completion of the pure universal completion of  $P$ .*

Notice that Theorem 6.1 highlights the universal properties of the Dialectica construction, providing a useful tool to define new connections between doctrines. We employ this characterisation to establish relationships between Dialectica doctrines and the doctrines for computability we discussed already, Medvedev, and Weihrauch.

We start by considering the Medvedev doctrine  $\mathfrak{M}: \text{Set}^{\text{op}} \longrightarrow \text{Pos}$  as presented in Definition 4.3.

<sup>3</sup>See also the fibrational setting in [TSdP21].

**Theorem 6.2.** *Let  $\mathfrak{M}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  be the Medvedev doctrine. Then there exists a morphism of existential and universal doctrines*

$$\mathfrak{E}: \mathfrak{Dial}(\mathbb{A}_M) \rightarrow \mathfrak{M}$$

such that the diagram

$$\begin{array}{ccc} & & \mathfrak{Dial}(\mathbb{A}_M) \\ & \nearrow \eta_{\mathbb{A}_M}^{\exists\forall} & \downarrow \mathfrak{E} \\ \mathbb{A}_M & \xrightarrow{\eta_{\mathbb{A}_M}^{\forall f}} & \mathfrak{M} \end{array}$$

commutes.

*Proof.* By Proposition 4.5 we have that the Medvedev doctrine  $\mathfrak{M}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  is full universal, hence it is in particular universal. Therefore, by the universal properties of the pure universal completion, we have that there exists a morphism of universal doctrines such that the diagram

$$\begin{array}{ccc} & & (\mathbb{A}_M)^{\forall} \\ & \nearrow \eta_{\mathbb{A}_M}^{\forall} & \downarrow \mu \\ \mathbb{A}_M & \xrightarrow{\eta_{\mathbb{A}_M}^{\forall f}} & \mathfrak{M} \end{array}$$

commutes. Now, by Proposition 4.5 we have that  $\mathfrak{M}: \mathbf{Set}^{\text{op}} \longrightarrow \mathbf{Pos}$  is also existential, hence we can apply the universal property of the pure existential completion, obtaining the existence of an arrow such that

$$\begin{array}{ccc} & & ((\mathbb{A}_M)^{\forall})^{\exists} \\ & \nearrow \eta_{\mathbb{A}_M}^{\exists} & \downarrow \mathfrak{E} \\ (\mathbb{A}_M)^{\forall} & \xrightarrow{\mu} & \mathfrak{M} \end{array}$$

Employing Theorem 6.1, we have that  $((\mathbb{A}_M)^{\forall})^{\exists} \equiv \mathfrak{Dial}(\mathbb{A}_M)$ , hence, combining the previous two diagrams, we can conclude that there exists an arrow such that the diagram

$$\begin{array}{ccc} & & \mathfrak{Dial}(\mathbb{A}_M) \\ & \nearrow \eta_{\mathbb{A}_M}^{\exists\forall} & \downarrow \mathfrak{E} \\ \mathbb{A}_M & \xrightarrow{\eta_{\mathbb{A}_M}^{\forall f}} & \mathfrak{M} \end{array}$$

commutes. □

Similarly, we can provide a connection with Weihrauch doctrines by employing the universal property of pure existential and pure universal completions.

**Theorem 6.3.** *Let  $\mathfrak{W}: \text{Mod}^{\text{op}} \longrightarrow \text{Pos}$  be the Weihrauch doctrine in Definition 5.6. Then there exists a morphism of existential doctrines*

$$\mathfrak{S}_1: \mathfrak{W} \rightarrow \text{Dial}(\mathfrak{e}\mathfrak{W})$$

and a morphism of existential and universal doctrines

$$\mathfrak{S}_2: \text{Dial}(\mathfrak{e}\mathfrak{W}) \rightarrow \mathfrak{W}$$

such that the diagram

$$\begin{array}{ccc}
 & & \mathfrak{W} \\
 & \nearrow \eta_{\mathfrak{e}\mathfrak{W}}^{\exists} & \uparrow \mathfrak{S}_2 \\
 \mathfrak{e}\mathfrak{W} & \xrightarrow{\eta_{\mathfrak{e}\mathfrak{W}}^{\forall}} & \text{Dial}(\mathfrak{e}\mathfrak{W}) \\
 & & \downarrow \mathfrak{S}_1
 \end{array}$$

commutes, and such that  $\mathfrak{S}_2\mathfrak{S}_1 \cong \text{id}_{\mathfrak{W}}$ .

**Remark 6.4.** Theorem 6.2 and Theorem 6.3 provide a formal connection between reducibility-like notions and dialectica interpretation, employing the universal properties of quantifiers completions. In particular, Theorem 6.3 provides a strong connection by showing that Weihrauch doctrines are full and reflective subdoctrines of dialectica doctrines, while Theorem 6.2 provides a canonical morphism from dialectica doctrines into Medvedev doctrines, that is not reflective in general. Notice that this difference is due to the different nature of Weihrauch and Medvedev doctrines, since one is an instance of pure existential completion, while the other is an instance of full universal completion.

## 7. CONCLUSIONS

We set out to and managed to categorify the notions of Medvedev, Muchnik and Weihrauch reducibility.

To show the categorification works, we proved the Medvedev isomorphism theorem, the Muchnik isomorphism theorem and the Weihrauch isomorphism theorems. We also showed how the respective Medvedev, Muchnik and Weihrauch doctrines relate to existential and universal completions. Lastly, using the doctrines quantifier properties, we showed how the Dialectica doctrine relates to the Medvedev and to the Weihrauch doctrines. This justifies our claims that bringing categorical tools into computable reducibility relations clarifies and connects the respective notions of reducibility, strengthening the connections with realizability theories.

But many questions remain. We would like to be able to introduce Turing reducibility into this generic picture, if possible. We also would like to understand better whether the work on choice principles in [BGP21] can be related to the work on choice in existential completions [MPR17]. Finally Miquel [Miq20] introduced the notion of an **implicative tripos**, that is, a tripos associated to an implicative algebra. The notion of implicative tripos encompasses all realizability triposes, classical realizability triposes, forcing triposes and so on. It would be good to understand if and how the notions of implicative algebra and

implicative tripos are related to Weihrauch doctrines. In particular, implicative triposes are defined over sets. Can we define an implicative-like tripos on the category of modest sets? If this is the case, does this notion encompass the notion of Weihrauch reducibility?

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