Contextual Constructive Description Logics

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Abstract

Constructive modal logics come in several different flavours and constructive *description* logics, while much more recent and less studied, not surprisingly, do the same. After all, it is a well-known result of Schild that description logics are simply variants of n-ary basic modal logic. There are several extensions of classical description logics, with modalities, temporal assertions, etc. As far as we know there are no such extensions for constructive description logics. Hence this note is a formal description of the extension of a constructive description logic c$\mathcal{ALC}$ [MS08] with contexts as modalities, as described in [deP03a], following the blueprint of [WZ99].

Keywords: constructive logic, description logic, contexts

1 Introduction

Description Logics are a knowledge representation formalism, much used in Artificial Intelligence. They are logic-based formalisms intended for representing knowledge about concept hierarchies, supplied with effective reasoning procedures and a declarative semantics. Description logics are very popular nowadays, perhaps due to their proposed applications in the Semantic Web. Most uses of description logics consider classical systems. However, considering versions of *constructive* description logics makes sense, both from a theoretical and from a practical viewpoint, as discussed in [deP03].

Description logics tend to be bundled in families of logical systems, depending on which concept *constructors* you allow in the logic. Since description logics came into existence as fragments of first-order logic, chosen to find the best trade-off possible between expressiveness and tractability of the fragment, several systems were discussed and eventually a taxonomy of systems emerged. In this taxonomy, the system called ALC (for Attributive Language with Complements) has come to be known as the canonical basic one. As far as *constructive* description logics are concerned, Mendler and Scheele have worked out a very compelling system, which they call $c\mathcal{ALC}$ ([MS08], based on the constructive modal logic CK[BPR01]). A different constructive version of ALC, based on the framework for constructive modal logics developed by Simpson in his phd thesis [Sim95] was developed by Hausler, Rademaker and de Paiva. Their system, called iALC for Intuitionistic ALC, was described in [HRP10] and it is the reduced version of Braüner and de Paiva's system of Intuitionistic Hybrid logic, IHL [BdP06]. The systems $c\mathcal{ALC}$ and iALC are alternative formalizations of constructive description logics, and the main difference between these systems is whether they satisfy (or not) distribution of possibility over disjunction.

In this note, we start by recalling the description logic $c\mathcal{ALC}$. We then consider one extra modality on top of $c\mathcal{ALC}$, following the blueprint of [WZ99], and prove decidability of the resulting system. In previous work [deP03a] one of us suggested the use of constructive modalities as MacCarthy-style contexts in AI. On that work, the application envisaged was constructive modalities for formalising natural language 'microtheories' and it was regretted that the system obtained only described constructive modalities over a propositional basis. The work in this paper gets closer to the desired ultimate system, as we can now talk about context as a constructive modality over a constructive description logic basis. But most of the hard work is still to be done, as it concerns the interactions of the finite (but very large) collection of linguistics based contexts/modalities. Here we briefly discuss this as future work.

2 Constructive description logic $c\mathcal{ALC}$

The basic building blocks of description logics are *concepts*, *roles* and *individuals*. We think of concepts as unary predicates in usual first-order logic and of roles as binary predicates, used to modify the concepts. Like classical ALC [DL03] the intuitionistic version c$\mathcal{ALC}$ is a basic description language whose concept constructors are described by the following grammar:

$$C, D ::= A \mid \bot \mid \top \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid C \sqsubseteq D \mid \exists R.C \mid \forall R.C$$

where A stands for an atomic concept and R for an atomic role. Note however that as in [MS08] we treat $\sqsubseteq$ as a concept forming operator, unlike in ALC. In the classical setting, $C \sqsubseteq D$ would have been definable as $\neg C \sqcup D$, but we are in the constructive setting and connectives are not interdefinable.

Following Mendler and Scheele we say a constructive interpretation of c$\mathcal{ALC}$ is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$

consisting of a non-empty set $\Delta^{\mathcal{I}}$ of entities in which each entity represents a partially defined individual; a refinement pre-ordering $\preceq^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$, i.e., a reflexive and transitive relation; $\perp^{\mathcal{I}}$ is a subset of fallible entities satisfying $\perp$ (fallible entities are over-defined and hence self-contradictory. This set is closed under refinement, that is, $x \in \perp^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in \perp^{\mathcal{I}}$); and an interpretation function $\cdot^{\mathcal{I}}$ mapping each role name R to a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and each atomic concept A to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ which is closed under refinement, i.e., $x \in A^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in A^{\mathcal{I}}$. The interpretation $\mathcal{I}$ is lifted from atomic $\perp, A$ to arbitrary concepts, where $\Delta_c^{\mathcal{I}} =_{df} \Delta^{\mathcal{I}} \setminus \perp^{\mathcal{I}}$ is the set of non-fallible elements, via:

$$\begin{aligned}
\top^{\mathcal{I}} &=_{df} \Delta^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta_c^{\mathcal{I}}. x \preceq y \Rightarrow y \notin C^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \sqsubseteq D)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta_c^{\mathcal{I}}. (x \preceq y \text{ and } y \in C^{\mathcal{I}}) \Rightarrow \\
&\qquad y \in D^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta_c^{\mathcal{I}}. x \preceq y \Rightarrow \\
&\qquad \exists z \in \Delta^{\mathcal{I}}. (y,z) \in R^{\mathcal{I}} \text{ and } z \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} &=_{df} \{x | \forall y \in \Delta_c^{\mathcal{I}}. x \preceq y \Rightarrow \\
&\qquad \forall z \in \Delta^{\mathcal{I}}. (y,z) \in R^{\mathcal{I}} \Rightarrow z \in C^{\mathcal{I}}\}
\end{aligned}$$

Semantic validity can be introduced as follows: say "x satisfies C in the interpretation $\mathcal{I}$", written as $\mathcal{I}, x \models C$, if x is in the interpretation of C, $x \in C^{\mathcal{I}}$. Say $\mathcal{I} \models C$ if this happens for all x in $\Delta^{\mathcal{I}}$. Finally say $\models C$ if for all interpretations $\mathcal{I}$ we have $\mathcal{I} \models C$. These definitions are usually extended to sets of concepts.

Typical reasoning in description logics is done via TBoxes and ABoxes. If we use Θ for a TBox, i.e., a collection of concepts and subsumptions[1] and Γ for an ABox, a collection of instantiations of concepts then we can say $\Theta, \Gamma \models C$ if for all interpretations $\mathcal{I}$, which are models of all the concepts in Θ it is the case that every x in $\mathcal{I}$ which satisfy the axioms in Γ must also satisfy C, or

$$\forall \mathcal{I}. \forall x \in \Delta^{\mathcal{I}}. (\mathcal{I} \models \Theta \text{ and } \mathcal{I}, x \models \Gamma) \text{ implies } \mathcal{I}, x \models C$$

Note that if we only consider TBox reasoning, that is if the ABox is empty, the definition above gives $\Theta, \emptyset \models C$ iff

$$\forall \mathcal{I}. \forall x \in \Delta^{\mathcal{I}}. \, \mathcal{I} \models \Theta \text{ implies } \mathcal{I}, x \models C$$

A Hilbert-style axiomatization of TBox reasoning in c$\mathcal{ALC}$ consists of the axioms and rules given in Figure 1. We denote derivability in this caclulus as $\vdash_H$.

Mendler and Scheele([MS08] p.7) proved:

Theorem 1 (Mendler-Scheele)**.** *The Hilbert calculus given in Figure 1 is sound and complete for TBox reasoning, that is* $\Theta, \emptyset \models C$ *if and only if* $\Theta \vdash_H C$.

[1] This is a somewhat non-traditional usage, since usually a TBox would contain subsumptions, rather than concepts. However recall that following [MS08], we express subsumptions as concepts.

$$\begin{array}{rc}
(\sqsubseteq_1) & C \sqsubseteq (D \sqsubseteq C) \\
(\sqsubseteq_2) & (C \sqsubseteq (D \sqsubseteq E)) \sqsubseteq ((C \sqsubseteq D) \sqsubseteq (C \sqsubseteq E)) \\
(\sqcap_1) & C \sqcap D \sqsubseteq C \\
(\sqcap_2) & C \sqcap D \sqsubseteq D \\
(\sqcap_3) & C \sqsubseteq (D \sqsubseteq (C \sqcap D)) \\
(\sqcup_1) & C \sqsubseteq C \sqcup D \\
(\sqcup_2) & D \sqsubseteq C \sqcup D \\
(\sqcup_3) & (C \sqsubseteq E) \sqsubseteq ((D \sqsubseteq E) \sqsubseteq (C \sqcup D \sqsubseteq E)) \\
(\perp) & \perp \sqsubseteq C \\
(\forall K) & \forall R.(C \sqsubseteq D) \sqsubseteq (\forall R.C \sqsubseteq \forall R.D) \\
(\exists K) & \forall R.(C \sqsubseteq D) \sqsubseteq (\exists R.C \sqsubseteq \exists R.D) \\
(Nec) & \text{If } \vdash_H C \text{ then } \vdash_H \forall R.C \\
(MP) & \text{If } \vdash_H C \text{ and } \vdash_H C \sqsubseteq D, \text{ then} \vdash_H D
\end{array}$$

Figure 1: The System c$\mathcal{ALC}$: Hilbert-style

A sequent calculus version of c$\mathcal{ALC}$ is given by Mendler and Scheele, who also prove cut-elimination for their calculus. But their calculus is a a tableau style calculus with positive and negative information about concepts and a simpler one is available, see Figure 2.

Note that our version, which is constructive, has restrictions to a single conclusion formula in the rules for subsumption and universal-quantification-role on the right, which are essential to keep the system intuitionistic. It is reassuring to see the same rules for roles in Straccia's 4-valued Description Logic [Str97]. The rules for the propositional connectives ($\sqcap, \sqcup$) are basically the same as for classical ALC, and the rules for subsumption $\sqsubseteq$ are just the rules for intuitionistic implication.

The system c$\mathcal{ALC}$[MS08] is related to constructive CK ([BPR01] and [MdP05]) in the same way classical multimodal K is related ALC[Sch91]. In the system c$\mathcal{ALC}$, the classical principles of the excluded middle $C \sqcup \neg C = \top$, double negation elimination $\neg\neg C = C$ and the definitions of the modalities $\exists R.C = \neg \forall R. \neg C$ and $\forall R.C = \neg \exists R. \neg C$ are no longer tautologies, but simply non-trivial TBox statements used to axiomatize specific application scenarios.

Soundness and completeness of a sequent calculus version of c$\mathcal{ALC}$is indicated in page 10 of [MS08], although not exactly for the sequent calculus we proposed in Figure 2. Our sequents are simpler than theirs, as we do not insist in carrying negative information along derivations, as they do. Nonetheless we have:

Theorem 2. *The sequent calculus for c$\mathcal{ALC}$ in Figure 2 and the Hilbert calculus described in Figure 1 are equivalent. For any TBox* Θ *and concept* C*, we have that* $\Theta, \emptyset \vdash_H C$ *if and only if the sequent* $\Theta \Rightarrow C$ *has a derivation using the rules in Figure 2.*

The proof of soundness and completeness of the sequent calculus for c$\mathcal{ALC}$ does not come straight from Straccia's work, as our rules for roles are the same, but our semantics

$$\frac{}{\Gamma, C \Rightarrow C, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow C, \Delta}$$

$$\frac{\Gamma \Rightarrow C, \Delta \qquad \Gamma, D \Rightarrow \Delta}{\Gamma, C \sqsubseteq D \Rightarrow \Delta} \sqsubseteq\text{-l} \qquad \frac{\Gamma, C \Rightarrow D}{\Gamma \Rightarrow C \sqsubseteq D} \sqsubseteq\text{-r}$$

$$\frac{\Gamma, C, D \Rightarrow \Delta}{\Gamma, (C \sqcap D) \Rightarrow \Delta} \sqcap\text{-l} \qquad \frac{\Gamma \Rightarrow C, \Delta \qquad \Gamma \Rightarrow D, \Delta}{\Gamma \Rightarrow (C \sqcap D), \Delta} \sqcap\text{-r}$$

$$\frac{\Gamma, C \Rightarrow \Delta \qquad \Gamma, D \Rightarrow \Delta}{\Gamma, (C \sqcup D) \Rightarrow \Delta} \sqcup\text{-l} \qquad \frac{\Gamma \Rightarrow C, D, \Delta}{\Gamma \Rightarrow (C \sqcup D), \Delta} \sqcup\text{-r}$$

$$\frac{\Gamma, C \Rightarrow \Delta}{\Gamma, \forall R.C \Rightarrow \Delta} \forall\text{-l} \qquad \frac{\Gamma \Rightarrow C}{\Gamma \Rightarrow \forall R.C} \forall\text{-r}$$

$$\frac{\Gamma, C \Rightarrow \Delta}{\Gamma, \exists R.C \Rightarrow \Delta} \exists\text{-l} \qquad \frac{\Gamma \Rightarrow C, \Delta}{\Gamma \Rightarrow \exists R.C, \Delta} \exists\text{-r}$$

Figure 2: The System c$\mathcal{ALC}$: Sequent calculus

are different. Straccia insists on 4-valuedness, we only want constructiveness.

Theorem 3. *The sequent calculus described in Figure 2 is sound and complete for TBox reasoning, that is $\Theta, \emptyset \models C$ if and only if $\Theta \vdash_S C$.*

3 Extending c$\mathcal{ALC}$

The main idea of the extension of c$\mathcal{ALC}$with a constructive context operator $\Box$ from the logic CK [MdP05] is similar to the approach of Wolter and Zakharyaschev to modalizing classical description logics. Intuitively, we introduce a Kripke-style model to interpret the $\Box$ where each possible world is a c$\mathcal{ALC}$ model. We treat c$\mathcal{ALC}$ formulas as atomic formulas of the extended logic c$\mathcal{ALC}_\Box$.

If ϕ^{at} is a formula of c$\mathcal{ALC}$, then formulas of c$\mathcal{ALC}_\Box$are defined as follows:

$$\phi ::= \top \mid \bot \mid \phi^{at} \mid \Box\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi$$

The logic CK (see for example [MdP05]) is interpreted on models $(W, \leq, R, I)$ where W is a non-empty set of possible worlds, $\leq$ is a reflexive transitive binary relation on W, R is an arbitrary binary relation on W, and I is an interpretation function (p^I is a subset of W satisfying p). Inconsistent worlds are allowed, namely $\bot^I$ is not necessarily empty, so we have fallible worlds. The conditions on models are as follows:

- $\leq$ is hereditary with respect to atomic formulas, that is for every atomic p, if $w \in p^I$ and $w \leq w'$, then $w' \in p^I$ In particular, if $w \in \bot^I$ and $w \leq w'$, then $w' \in \bot^I$.
- if $w \in \bot^I$, then $w \in p^I$ for every atomic p.

A model of c$\mathcal{ALC}_\Box$, $\mathcal{M}$, is a CK-model $(W, \leq, R, I)$ where in addition each W is a c$\mathcal{ALC}$ model and for every formula ϕ^{at} of c$\mathcal{ALC}$, $w \in (\phi^{at})^I$ iff $w \models \phi^{at}$.

Definition 1 (satisfaction in $\mathcal{M}$)**.** *The relation "the c$\mathcal{ALC}_\Box$-model $\mathcal{M}$ and the world $w \in W$ satisfy a formula ϕ" (in symbols $\mathcal{M}, w \models \phi$) is defined inductively as follows:*

$\mathcal{M}, w \models \phi^{at}$ *iff* $w \in (\phi^{at})^I$

$\mathcal{M}, w \models \phi \wedge \psi$ *iff* $\mathcal{M}, w \models \phi$ *and* $\mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \phi \vee \psi$ *iff* $\mathcal{M}, w \models \phi$ *or* $\mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \phi \rightarrow \psi$ *iff for all* w' *with* $w \leq w'$ *if* $\mathcal{M}, w' \models \phi$, *then* $\mathcal{M}, w' \models \psi$

$\mathcal{M}, w \models \Box\phi$ *iff for all* w' *with* $w \leq w'$, $\forall u(R(w', u) \Rightarrow \mathcal{M}, u \models \phi)$

Theorem 4. *Satisfiability for c$\mathcal{ALC}_\Box$is decidable.*

Proof sketch. The decision procedure builds on decidability results for c$\mathcal{ALC}$ [MS08] and CK [MdP05], respectively. We formulate the procedure as a non-deterministic algorithm (guess a model and check that it is indeed a satisfying model) but it can also be described as a deterministic algorithm working by exhaustive enumeration of all models of fixed bounded size.

Given a formula ϕ of c$\mathcal{ALC}_\Box$, we first use the result from [MS08] to guess a bounded size CK model M for ϕ (ignoring the extra conditions for the c$\mathcal{ALC}_\Box$models). Note that for each world of M, by construction from [MdP05], only the interpretation of subformulas of ϕ in M matters. Let us call this set of subformulas $Sf(\phi)$. Note that $Sf(\phi)$ is finite. Now we check, for each world w of M, whether the set $\{\phi^{at} : M, w \models \phi^{at}$ and $\phi^{at} \in Sf(\phi)\}$ has a c$\mathcal{ALC}$ model. By the result of [MS08], this is decidable. If every world in M has a corresponding c$\mathcal{ALC}$ model, we are done: we found a c$\mathcal{ALC}_\Box$ model for ϕ. $\Box$

Having obtained decidability for c$\mathcal{ALC}_\Box$ we conjecture that the system can be extended with many non-interacting boxes, to provide a system $c\mathcal{ALC}_{ctx}$ for Artificial Intelligence contexts, the application to Natural Language semantics that we are after, which is briefly described below.

In a series of papers ([deP05],[Bal07],[Bal07a]) the PARC team has described a formalization of the semantics produced by their system Bridge, that automatically creates semantics for natural language sentences from the sentences themselves, using symbolic and statistical methods. The logic that emerges from their representations (which they called TIL for textual inference logic) is a description logic of concepts and contexts, determined by the lexical semantics of the words used. Their description corresponds intuitively to a system of constructive description logic, where the linguistic contexts are induced by intensional concepts, such as propositional attitude verbs and negations. For a trivial example consider the sentence *Ed knew that he closed the door*, in the logic TIL the representation of this sentence will have two contexts, one corresponding to the world of the things that Ed knew, one corresponding to how the author of the sentence conceives the real world to be. These linguistically induced contexts were also called 'microtheories', by analogy to CYC's contexts, which correspond to first-order logic theories. The work in this paper started from a desire to give a firm logical footing to the system TIL.

However, the PARC team also did some work in the interactions amongst their linguistic contexts [NCK06]. The ultimate goal of the research reported in this paper is to formalize that latter work on linguistics contexts interaction. As discussed by Nairn et al these interactions are predictable, but not trivial to formalize. Much work remains to be done towards this goal and we are barely starting it. However, it must be noted that we do not envisage adding any work along the lines of the work of Klarman and Gutierrez [KG10]. Despite the superficial similarity between our paper and their work (both papers add contexts MacCarthy-style to the description logic $\mathcal{ALC}$), the resulting systems are very different. First because what we take as contexts MacCarthy-style is different and secondly because of our commitment to make our logic constructive, both in the basis and in the context structure over it. It is hoped that the constructivity of the logic will 'pay' for itself via the usefulness of the Curry-Howard correspondence, as argued, *inter alia*, by Mendler and Scheele [MS08].

4 Conclusions

We extended the constructive description logic c$\mathcal{ALC}$ with a modality box operator and proved that the resulting logic is decidable. This extension is motivated by a proposed application to modelling contexts in AI, as described, in the propositional setting, in [deP03a]. Much remains to be done, in particular we want to investigate the complexity price of constructivity in our setting and we also must check the adaptation needs of our application.

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