# Dialectica Categories Revisited

Valeria de Paiva

Chapman University

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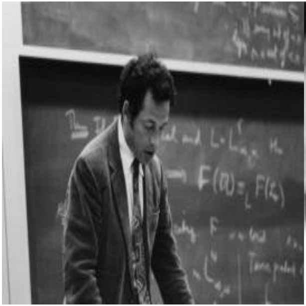
# Thanks!

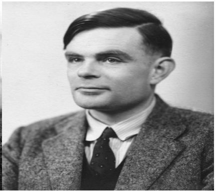
Marco and Gisele for the invitation today!

Sol and Grisha for the first invitation to Stanford, a long time ago.



# Personal stories

# Personal stories

# Personal stories


*"There are two ways to do great mathematics. The first way is to be smarter than everybody else. The second way is to be stupider than everybody else – but persistent." – Raoul Bott*

# Personal stories

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# Dialectica Interpretation

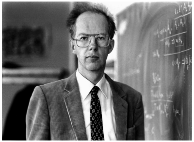
**Dialectica Interpretation (Gödel 1958):** an interpretation of intuitionistic arithmetic $\mathrm{HA}$ in a quantifier-free theory of functionals of finite type **System $\mathbf{T}$**.

**Idea:** translate every formula $A$ of $\mathrm{HA}$ to

$$A^D = \exists u \forall x A_D$$

where $A_D$ is quantifier-free.

# Dialectica Interpretation



**Application (Gödel 1958):** if $\mathrm{HA}$ proves $A$, then System $\mathrm{T}$ proves $A_D(t, x)$, where $x$ is a string of variables for functionals of finite type, and $t$ a suitable sequence of terms (not containing $x$).

**Goal:** to be as **constructive** as possible, while being able to interpret all of classical Peano arithmetic (Troelstra).

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes.*, Dialectica, 12(3-4):280–287. (Translation in Gödel's Collected Works)

# Dialectica interpretation

$A_D(u;x)$ quantifier-free formula defined inductively:

$$
\begin{array}{lcl}
(P)_D & \equiv & P \ (P \text{ atomic}) \\
(A \wedge B)_D(u,v;x,y) & \equiv & A_D(u;x) \wedge B_D(v;y) \\
(A \vee B)_D(u,v,z;x,y) & \equiv & (z=0 \rightarrow A_D(u;x)) \wedge (z \neq 0 \rightarrow B_D(v;y)) \\
(A \rightarrow B)_D(f,F;u,y) & \equiv & A_D(u;Fuy) \rightarrow B_D(fu;y) \\
(\exists z A)_D(u,x;z) & \equiv & A_D(u;x) \\
(\forall z A)_D(f;y,z) & \equiv & A_D(fz;y)
\end{array}
$$

**Theorem (Dialectica Soundness, Gödel 1958)**

*Whenever a formula $A$ is provable in Heyting arithmetic then there exists a sequence of closed terms $t$ such that $A_D(t;y)$ is provable in system $T$. The sequence of terms $t$ and the proof of $A_D(t;y)$ are constructed from the given proof of $A$ in Heyting arithmetic.*

# Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(A \to B)^D$

$$(A \to B)^D = \exists f, F \forall u, y (A_D(u, F(u, y)) \to B_D(f(u), y)).$$

**Intuition:** Given a witness $u$ in $U$ for the hypothesis $A_D$, there exists a function $f$ assigning a witness $f(u)$ to $B_D$. Moreover, from a counterexample $y$ to the conclusion $B_D$, we should be able to find a counterexample $F(u, y)$ for the hypothesis $A_D$.

Feferman et al editors (1986), *Kurt Gödel: Collected Works: Volume II*, Oxford University Press.

# Dialectica interpretation

Troelstra (p 226 Collected Works Gödel ) from Spector (1962)

$$[\exists u \forall x . A_D(u,x) \to \exists v \forall y . B_D(v,y)] \leftrightarrow^{(i)}$$

$$[\forall u (\forall x A_D(u,x) \to \exists v . \forall y (B_D(v,y))] \leftrightarrow^{(ii)}$$

$$[\forall u \exists v (\forall x . A_D(u,x) \to \forall y B_D(v,y))] \leftrightarrow^{(iii)}$$

$$[\forall u \exists v \forall y (\forall x A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(iv)}$$

$$[\forall u \exists v \forall y \exists x (A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(v)}$$

$$\exists V, X \forall u, y (A_D(u, X(u,y)) \to B_D(V(u), y))$$

where $(i)$ and $(iii)$ are intuitionistic, but $(ii)$ requires **Independence of Premise**, $(iv)$ requires **Markov Principle** and $(v)$ requires two uses of the **axiom of choice**.

# Dialectica interpretation

Hence translation involves three logical, non-intuitionistic, principles:

1. **Principle of Independence of Premise** (IP)

$$(A \to \exists v.B(v)) \to \exists v.(A \to B(v))$$

2. a generalisation/modification of **Markov Principle** (MMP)

$$(\forall x.A(x) \to B(y)) \to \exists x.(A(x) \to B(y))$$

3. the **axiom of choice** (AC)

$$\forall y.\exists x.A(x,y) \to \exists V.\forall y.A(V(y),y)$$

# Categorical Dialectica Construction

**Dialectica category (de Paiva 1988):** Given a category $C$ with finite limits, one can build a new category $\mathfrak{Dial}(\mathrm{C})$, whose objects have the form $A = (U, X, \alpha)$ where $\alpha$ is a subobject of $U \times X$ in $C$; **think** of this object as representing the formula

$$\exists u \forall x \alpha(u, x).$$

A map from $\exists u \forall x \alpha(u, x)$ to $\exists v \forall y \beta(v, y)$ can be thought of as a pair $(f : U \to V, F : U \times Y \to X)$ of terms/maps, subject to the entailment condition

$$\alpha(u, F(u, y)) \vdash \beta(f(u), y).$$

(First internalisation of the Dialectica interpretation!)

# Original Dialectica Constructions

Thesis: 4 chapters, 4 main theorems.
All of them of the form:
Category C is a categorical model of logic *L*.

all start from C cartesian closed cat + coproducts + (...)

Thm 1: $\mathfrak{Dial}$(C) is a model of !-free ILL

Thm 2: $\mathfrak{Dial}$(C) + ! (where ! is a co-free monoidal comonad) is a model of IL

Thm 3: Gir(C) (a simpler dialectica cat) is a model of (!,?)-free CLL/FILL

Thm 4: Gir(C) + !,? (!,? given by a composite monoidal (co)monad) is a model of IL/CL

# This Talk: only the first half

Only 2 main theorems:

Start with C a cartesian closed cat + coproducts + (...)

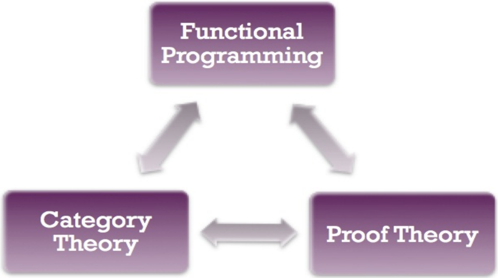
Apply Dialectica construction to it get to $\mathfrak{Dial}(C)$

Thm 1: $\mathfrak{Dial}(C)$ is a model of !-free Intuitionistic Linear Logic

Thm 2: $\mathfrak{Dial}(C)$ + !, where ! is a co-free monoidal comonad, is a model of $IL_{\to,\wedge}$ or simply typed lambda-calculus

**Why this is interesting?**

# Categorical Models

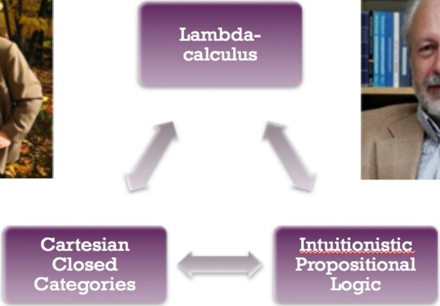



Types are formulae/objects in appropriate category,
Terms/programs are proofs/morphisms in the category,
Logical constructors are 'appropriate' categorical constructions.
Most important: Reduction is proof normalization (**Tait**)
Outcome: Transfer results/tools from logic to CT to CScience

# + Curry-Howard Correspondence

1963

1965



# Linear Logic

A proof theoretic logic described by Jean-Yves Girard in 1986.

Basic idea: assumptions cannot be discarded or duplicated. They must be used exactly once – just like dollar bills

Other approaches to accounting for logical resources before.

Win of Linear Logic: Account for resources when you want to, otherwise fall back on traditional logic, **Girard Translation**

$$A \to B \text{ iff } !A \multimap B$$

# Resources in Linear Logic

In Linear Logic formulas denote resources. Resources are premises, assumptions and conclusions, as they are used in logical proofs. For example:

$\$1 \multimap \text{latte}$

If I have a dollar, I can get a Latte

$\$1 \multimap \text{cappuccino}$

If I have a dollar, I can get a Cappuccino

$\$1$

I have a dollar

Using my dollar premise and one of the premises above, say '$\$1 \multimap \text{latte}$' gives me a latte but the dollar is gone
Usual logic doesn't pay attention to uses of premises, $A$ implies $B$ and $A$ gives me $B$ but I still have $A$

# Linear Implication and (Multiplicative) Conjunction

Traditional implication: $A, A \to B \vdash B$

$A, A \to B \vdash A \wedge B$ Re-use $A$

Linear implication: $A, A \multimap B \vdash B$

$A, A \multimap B \nvdash A \otimes B$ Cannot re-use $A$

Traditional conjunction: $A \wedge B \vdash A$ Discard $B$

Linear conjunction: $A \otimes B \nvdash A$ Cannot discard $B$

Of course: $!A \vdash !A \otimes !A$ Re-use

$!A \otimes B \vdash I \otimes B \cong B$ Discard

# Challenges of modeling Linear Logic

Traditional categorical modeling of intuitionistic logic

formula $A \rightsquigarrow$ object $A$ of appropriate category

$A \wedge B \rightsquigarrow A \times B$ (real product)

$A \to B \rightsquigarrow B^A$ (set of functions from $A$ to $B$)

These are real products, so we have projections $(A \times B \to A, B)$ and diagonals $(A \to A \times A)$ which correspond to deletion and duplication of resources

**Not Linear!!!**
**Easy**: Need to use *tensor products* and *internal homs* in CT $\Rightarrow$ symmetric monoidal closed category
**Hard**: how to define the *make-everything-usual* operator "!"

# Dialectica Categories

Hyland suggested that to provide a categorical model of the Dialectica Interpretation, one should look at the functionals corresponding to the interpretation of logical implication.

I looked and instead of finding a cartesian closed category, found a monoidal closed one

Thus the categories in my thesis proved to be models of Linear Logic

# Category $\mathfrak{Dial}(C)$

Start with a cat $C$ that is cartesian closed with pullbacks. Then build a new category $\mathfrak{Dial}(()C)$.

Objects are relations in $C$, triples $(U, X, \alpha)$, $\alpha : U \times X \to 2$, so either $u\alpha x$ or not.

Maps are pairs of maps in $C$. A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of maps in $C$, $(f : U \to V, F : U \times Y \to X)$ such that a 'semi-adjunction condition' is satisfied: for $u \in U, y \in Y$, $u\alpha F(u, y)$ implies $fu\beta y$. (**Note direction and dependence!**)

**Theorem1: (de Paiva 1987) [Linear structure]**

The category $\mathfrak{Dial}$(C) has a symmetric monoidal closed structure (and products, weak coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

# Can we give some intuition for these objects?

Blass makes the case for thinking of problems in computational complexity. Samuel da Silva and I say you can think of **Kolmogorov-Veloso problems**. Many other interpretations make sense.

Intuitively an object of $\mathfrak{Dial}(C)$

$$A = (U, X, \alpha)$$

can be seen as representing a problem.

The elements of $U$ are instances of the problem, while the elements of $X$ are possible answers to the problem instances.

The relation $\alpha$ checks whether the answer is correct for that instance of the problem or not.

(Superpower games?)

# Examples of objects in $\mathfrak{Dial}(C)$

1. The object $(\mathbb{N}, \mathbb{N}, =)$ where $n$ is related to $m$ iff $n = m$.
2. The object $(\mathbb{N}^{\mathbb{N}}, \mathbb{N}, \alpha)$ where $f$ is $\alpha$-related to $n$ iff $f(n) = n$.
3. The object $(\mathbb{R}, \mathbb{R}, \leq)$ where $r_1$ and $r_2$ are related iff $r_1 \leq r_2$
4. The objects $(2, 2, =)$ and $(2, 2, \neq)$ with usual equality inequality.

# Tensor product in $\mathfrak{Dial}(\mathsf{C})$

Given objects $(U, X, \alpha)$ and $(V, Y, \beta)$ it is natural to think of $(U \times V, X \times Y, \alpha \times \beta)$ as a tensor product.

This construction does give us a bifunctor

$$\otimes \colon \mathfrak{Dial}(\mathsf{C}) \times \mathfrak{Dial}(\mathsf{C}) \to \mathfrak{Dial}(\mathsf{C})$$

with a unit $I = (1, 1, id_1)$.

Note that this is not a product.
There are no projections $(U \times V, X \times Y, \alpha \times \beta) \to (U, X, \alpha)$.
Nor do we have a diagonal functor
$\Delta \colon \mathfrak{Dial}(\mathsf{C}) \to \mathfrak{Dial}(\mathsf{C}) \times \mathfrak{Dial}(\mathsf{C})$, taking
$(U, X, \alpha) \to (U \times U, X \times X, \alpha \times \alpha)$

# Internal-hom in $\mathfrak{Dial}(C)$

To internalize the notion of map between problems, we need to consider the collection of all maps from $U$ to $V$, $V^U$, the collection of all maps from $U \times Y$ to $X$, $X^{U \times Y}$ and we need to make sure that a pair $f \colon U \to V$ and $F \colon U \times Y \to X$ in that set, satisfies the dialectica condition:

$$\forall u : U, y : Y, u\alpha F(u, y) \to fu\beta y$$

This give us an object in $\mathfrak{Dial}(C)$ $(V^U \times X^{U \times Y}, U \times Y, \beta^\alpha)$
The relation $\beta^\alpha \colon V^U \times X^{U \times Y} \times (U \times Y) \to 2$ evaluates a pair $(h, H)$ of maps on the pair of elements $(u, y)$ and checks the dialectica implication between the relations.

# Internal-hom in $\mathfrak{Dial}(C)$

Given objects $(U, X, \alpha)$ and $(V, Y, \beta)$ we can internalize the notion of morphism of $\mathfrak{Dial}(C)$ as the object $(V^U \times X^{U\times Y}, U \times Y, \beta^\alpha)$

This construction does give us a bifunctor, contravariant in the first coordinate and covariant in the second, as expected

The kernel of our first main theorem is the adjunction:

$$A \otimes B \to C \text{ if and only if } A \to [B \multimap C]$$

where $A = (U, X, \alpha), B = (V, Y, \beta)$ and $C = (W, Z, \gamma)$

# Products and Coproducts in $\mathfrak{Dial}(C)$

Given objects $(U, X, \alpha)$ and $(V, Y, \beta)$ it is natural to think of $(U \times V, X + Y, \alpha \circ \beta)$ as a categorical product in $\mathfrak{Dial}(C)$.

Since this is a relation on the set $U \times V \times (X + Y)$, either this relation has a $(x, 0)$ or a $(y, 1)$ element, and hence the $\circ$ symbol only 'picks' the correct relation $\alpha$ or $\beta$.

However, we do not have coproducts. It is only a **weak-coproduct** enough for the logic/type theory

**Theorem: (de Paiva 1987) [linear structure]**

The category $\mathfrak{Dial}(C)$ has a symmetric monoidal closed structure (and products, weak coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

# What about the Modality?

We need an operation on objects/propositions such that:

$!A \to !A \otimes !A$ (duplication)

$!A \to I$ (erasing)

$!A \to A$ (dereliction)

$!A \to !!A$ (digging)

Also ! should be a functor, i.e $(f, F) : A \to B$ then $!(f, F) : !A \to !B$

**Theorem: [linear and usual logic together]**

There is a **monoidal** comonad ! in $\mathfrak{Dial}$(C) which models exponentials/modalities and recovers Intuitionistic (and via DN Classical) Logic.

Take $!(U, X, \alpha) = (U, X^*, \alpha*)$, where $(-)^*$ is the free commutative monoid in $C$.

# (Cofree) Modality !

To show this works we need to show several propositions:

! is a monoidal comonad: there is a natural transformation $m_{(}-,-) :!A\otimes!B \to!(A \otimes B)$ and $m_I : I \to!I$ satisfying many comm diagrams

! induces a commutative comonoid structure on $!A$

$!A$ also has naturally a coalgebra structure induced by the comonad !

The comonoid and coalgebra structures interact nicely.

There are plenty of other ways to phrase these conditions. The more usual way nowadays is

**Theorem: [Linear and non-Linear logic together]**

There is a symmetric **monoidal** adjunction between $\mathfrak{Dial}(C)$ and its **cofree** coKleisli category for the monoidal comonad ! above.

# Cofree Modality !

Old way: "There is a monoidal comonad ! on a linear category $\mathfrak{Dial}(C)$ satisfying (lots of conditions)"and

**Theorem: [Linear and non-Linear (Benton) logic together]**

The coKleisli category associated with the comonad ! on $\mathfrak{Dial}(C)$ is cartesian closed.

To show cartesian closedness we need to show:

$$Hom_{Kl!}(A\&B, C) \cong Hom_{Kl!}(A, [B, C]_{Kl!})$$

The proof is then a series of equivalences that were proved before:
$Hom_{Kl!}(A\&B, C) \cong Hom_{DialC}(!(A\&B), C) \cong$
$Hom_{DialC}(!A \otimes !B, C) \cong Hom_{DialC}(!A, [!B, C]_{DialC}) \cong$
$Hom_{kl!}(A, [!B, C]_{DialC}) \cong Hom_{kl!}(A, [B, C]_{kl!})$
(Seely, 1989; de Paiva, 1989)

# What is the point of Dialectica categories?

The Dialectica construction provides a model of Linear Logic, instead of intuitionistic logic. This justifies LL in terms of a more traditional, proof-theoretic tool and conversely explains the more traditional work in terms of a 'modern' (linear, resource conscious) decomposition.

Dialectica categories are a *good* model of Linear Logic, as they keep the differences that Girard wanted to make. (see work with Andrea Schalk on L-valued models of LL).

This bolsters the claims about Curry-Howard and trinitarism, the connections to programming and CT as guidance.

Moreover, recent work with Trotta and Spadetto allows us to see where the assumptions in Gödel's argument (hacks?) are used.

# Other Categorical Dialectica Constructions

Most of the work in the original Dialectica categories (de Paiva 1989, 1991) was on the categorical structure needed to model Linear Logic (Girard 1987).

We described symmetric monoidal closed categories with appropriate (co)monads, modelling the modality !

This model is pretty cool! Lots of recent work on it, 30+ years later.
**Generalization:** initial construction has been generalized for arbitrary fibrations, by Hyland, Biering, Hofstra, von Glehn, Moss, etc.

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Hofstra (2011), *The dialectica monad and its cousins.*, A tribute to M. Makkai
Trotta, Spadetto and de Paiva (2021), *The Gödel fibration.*, MFCS 2021
Trotta, Spadetto and de Paiva (2022), *Gödel Doctrines.*, LFCS 2022

# Dialectica via Doctrines

Trotta, Spadetto and V. describe a categorical version of Dialectica in terms of (Lawvere's) **doctrines**.

In three arXiv preprints, two conference papers and two journal papers their work explains how modelling of the dialectica interpretation using doctrines (or fibrations) is very tight.

But **why** do we do it?

Isn't the modelling using categories enough?

Trotta, Spadetto and de Paiva (2023), *Dialectica principles via Gödel doctrines*, TCS
Trotta, Spadetto and de Paiva (2022), *Dialectica logical principles: not only rules.*, JLC 2022

# Dialectica via Doctrines

Two reasons:

1. First-order is of course more expressive than propositional logic, sometimes we need the extra expressivity;
2. Much tighter correspondence between the logic and the category theory, as exemplified by the Dialectica logical principles paper

In particular we get the ability to show how the internalisation of morphisms work.

# Dialectica via Doctrines

How well does the construction of the Dialectica categories (or doctrines) capture the essential ingredients of Gödel's original interpretation?

1. Given a doctrine $P$, when is there a doctrine $P'$ such that $\mathfrak{Dial}(P') \cong P$?

2. When such doctrine $P'$ exists, how do we find it?

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   Such a $P'$ exists precisely when $P$ is a **Gödel doctrine**
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   Such a $P'$ exists precisely when $P$ is a **Gödel doctrine**

2. When such doctrine $P'$ exists, how do we find it? $P'$ is given by the **quantifier-free elements** of the Gödel doctrine $P$

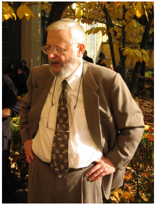
# Dialectica via Doctrines

As we saw the Dialectica translation requires some classical principles:
independence of premise(IP)
Markov principle (MP)
and the axiom of choice (AC).

How can we see these principles in our categorical modelling?

Can these categories and these principles be described in more conceptual terms, for example, in terms of universal properties?

# Doctrines



Lawvere defined hyperdoctrines, we start with less.

**Definition**

A **doctrine** is just a functor from a category $\mathcal{C}$ with finite products, to Pos, the category of posets.

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$$

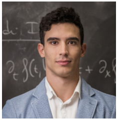
# Existential and Universal Doctrines

**Definition (existential/universal doctrines)**

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is ***existential*** (resp. ***universal***) if, for every $A_1$ and $A_2$ in $\mathcal{C}$ and every projection $A_1 \times A_2 \xrightarrow{\pi_i} A_i$, $i = 1, 2$, the functor:

$$PA_i \xrightarrow{P_{\pi_i}} P(A_1 \times A_2)$$

has a left adjoint $\exists_{\pi_i}$ (resp. a right adjoint $\forall_{\pi_i}$), and these satisfy the ***Beck-Chevalley conditions***.



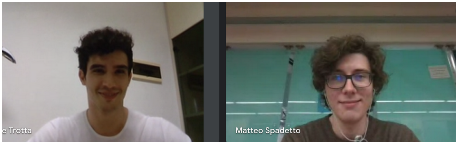
(Trotta (TAC 2020): The existential completion exists and satisfies all 2-categorical properties you may want. Ditto for the universal completion.)

# Doctrines and quantifier-free formulas

We want a suitable universal property to represent predicates that are quantifier-free, **categorically**. We have dual definitions for existential and universal quantifiers.

The paper defines:

- existential splitting predicates,
- existential-free predicates,
- doctrines $P$ with enough existential-free predicates.




Trotta, Spadetto and de Paiva (2023), *Dialectica principles via Gödel doctrines*, TCS

## Definition (Gödel doctrine)

A doctrine $P\colon \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$ is called a ***Gödel doctrine*** if:

1. the category $\mathcal{C}$ is cartesian closed;
2. the doctrine $P$ is existential and universal;
3. the doctrine $P$ has enough existential-free predicates;
4. the existential-free objects of $P$ are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_\pi(\alpha)$ is existential-free for every projection $\pi$ from $A$;
5. the sub-doctrine $P'\colon \mathcal{C}^{\text{op}} \longrightarrow \text{Pos}$ of the existential-free predicates of $P$ has enough universal-free predicates.

a mouthful! without item 5 we call it a ***Skolem doctrine***.

## Definition (Dialectica doctrine)

Let $P\colon \mathcal{C}^{op} \longrightarrow \text{Pos}$ be a doctrine whose base category C is cartesian closed. The **dialectica doctrine** $\mathfrak{Dial}(P)\colon \mathcal{C}^{op} \longrightarrow \text{Pos}$ is defined as the functor sending an object $I$ into the poset $\mathfrak{Dial}(P)(I)$ defined as follows:

**objects** are quadruples $(I, U, X, \alpha)$ where $I, X$ and $U$ are objects of the base category $\mathcal{C}$ and $\alpha \in P(I \times U \times X)$;

**partial order:** we say that $(I, U, X, \alpha) \leq (I, V, Y, \beta)$ if there exists a pair $(f_0, f_1)$, where $I \times U \xrightarrow{f_0} V$ and $I \times U \times Y \xrightarrow{f_1} X$ are morphisms of $\mathcal{C}$ such that:

$$\alpha(i, u, f_1(i, u, y)) \leq \beta(i, f_0(i, u), y).$$

This is a direct adaptation to the proof irrelevant setting of Hofstra's definition of Dialectica fibration.

**Theorem (Hofstra 2011)**

*If $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Pos}$ is a doctrine, then there is an isomorphism $\mathfrak{Dial}(P) \cong (P^{\forall})^{\exists}$ which is natural in $P$.*

(Here $Q^{\forall}$ and $Q^{\exists}$ denote the universal and the existential completions of any doctrine $Q$.)

**Theorem**

*Every Gödel doctrine $P$ is equivalent to the Dialectica completion $\mathfrak{Dial}(P')$ of the full subdoctrine $P'$ of $P$ consisting of the quantifier-free predicates of $P$.*

## Gödel doctrines in action

From now we can prove five theorems that justify our claim that the modelling provided by Gödel doctrines is very tight.

1. We can show that for a Gödel doctrine $P$ and any predicate $\alpha$ of $P(A)$, there exists a quantifier-free predicate $\alpha_D$ of $P(I \times U \times X)$ such that:

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U . \forall x : X . \alpha_D(i, u, x).$$

Thus in a Gödel doctrine every formula admits a presentation of the exact form used in the Dialectica translation.

2. We can show that morphisms of the dialectica categories correspond to implication in the Gödel doctrines.

# Gödel doctrines in action

3. We can that the skolemisation required by the Dialectica is modelled in Gödel doctrines.

For the next two theorems we need more than Gödel doctrines, we need (Lawvere's) hyperdoctrines, actually we say:

**Definition (Gödel hyperdoctrine)**

A hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ is called a ***Gödel hyperdoctrine*** when $P$ is a Gödel doctrine.

# Gödel doctrines in action

4. Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \text{Hey}$ satisfies the ***Rule of Independence of Premise***

5.Every Gödel hyperdoctrine $P\colon \mathcal{C}^{op} \longrightarrow \text{Hey}$ satisfies the following ***Modified Markov Rule***: whenever $\beta_D \in P(A)$ is a quantifier-free predicate and $\alpha \in P(A \times B)$ is an existential-free predicate, it is the case that:

$$a : A \mid \top \vdash (\forall b.\alpha(a, b)) \rightarrow \beta_D(a)$$

implies that

$$a : A \mid \top \vdash \exists b.(\alpha(a, b) \rightarrow \beta_D(a)).$$

## Summarizing

Used existential and universal doctrines (and their completions) to provide notions of quantifier-free formulae

Showed that the Gödel doctrines satisfy:
Dialectica Normal Form
Soundness of Implication
Skolemisation
Independence of Premise
Markov Principle

Obtained a very faithful categorical description of the Dialectica interpretation.

# Conclusions

Several categorical models of Gödel's Dialectica

Extended and generalized the original models.

Original models have several applications in logic (games, set theory), in functional and imperative programming, in concurrency, in automata theory. Pédrot and Kerjean say to differentiation too.

## Thank you!

***Elegant mathematics will of itself tell a tale, and one with the merit of simplicity. This may carry philosophical weight. But that cannot be guaranteed: in the end one cannot escape the need to form a judgement of significance.***
***Martin Hyland, 2004***

# Some References

(see https://github.com/vcvpaiva/DialecticaCategories)

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