Constructive Modal & Linear Logics

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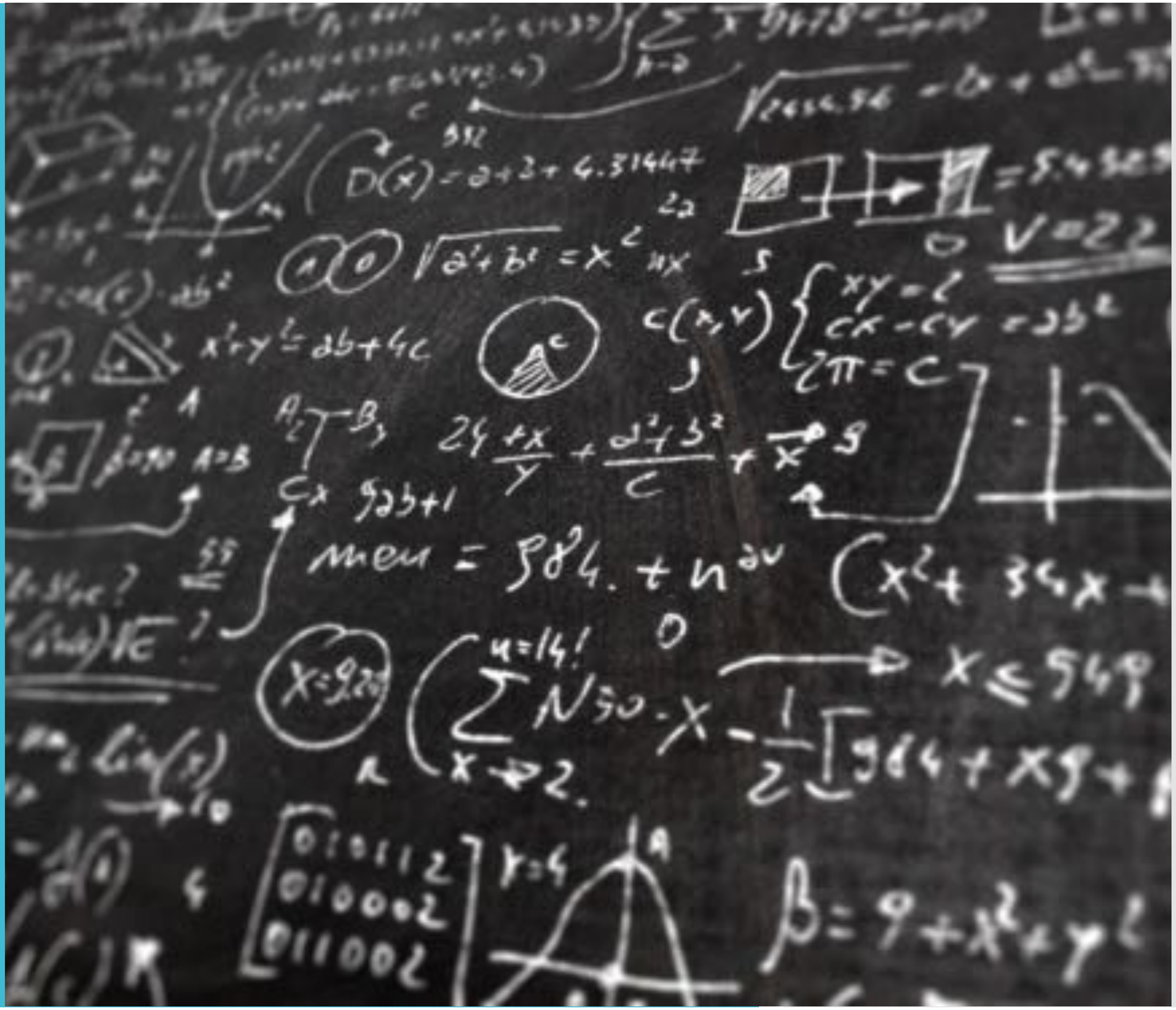


Thanks Shawn!

Logics

Frege: quantifiers

FOL

Hilbert: proofs as mathematical objects

axiom systems

Gentzen: sequents and Natural Deduction

Prawitz: normalization of Natural Deduction

Martin-Loef: type theory

Girard: system F

Linear Logic

Logics

- Frege: quantifiers & analytic philosophy 1879
- Hilbert: proofs as mathematical objects & axiom systems 1900
- Gentzen: sequents and Natural Deduction 1934
- Prawitz: normalization of Natural Deduction 1965
- **Martin-Löf**: type theory& dependent types 1972
- Girard: System F and Linear Logic 1987

Frege

quantifiers
analytic philosophy

Hilbert

Proofs as mathematical objects

Axiom systems

Hilbert

1. $A \supset (B \supset A)$
2. $(A \supset B \supset C) \supset ((A \supset B) \supset (A \supset C))$
3. $A \supset (B \supset A \wedge B)$
4. $A \wedge B \supset A$
5. $A \wedge B \supset B$
6. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
7. $A \supset A \vee B$
8. $B \supset A \vee B$
9. $\bot \supset A$

$$\frac{\Gamma \vdash A \supset B \qquad \Gamma \vdash A}{\Gamma \vdash B}\ \text{Modus Ponens}$$

$$(p \to (q \to p))$$

$$((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$$

$$((\neg p \to \neg q) \to (q \to p))$$

Gentzen

Sequent calculus

Natural Deduction

Gentzen

Sequent calculus

$$\frac{}{\Delta, A \vdash A}\ \text{Identity}$$

$$\frac{\Gamma \vdash B \qquad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \text{Cut}$$

$$\frac{}{\Gamma, \bot \vdash A}\ \bot\mathcal{L}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee\mathcal{L} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee\mathcal{R} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee\mathcal{R}$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\mathcal{L} \quad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge\mathcal{L} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\mathcal{R}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \supset B \vdash C}\ \supset\mathcal{L} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \supset\mathcal{R}$$

$$\frac{\bot}{A}\ \bot\mathcal{E}$$

$$\frac{\begin{matrix}[A^x] \\ \vdots \\ B\end{matrix}}{A \supset B}\ \supset\mathcal{I}_x \qquad \frac{A \supset B \quad A}{B}\ \supset\mathcal{E}$$

$$\frac{A \quad B}{A \wedge B}\ \wedge\mathcal{I} \qquad \frac{A \wedge B}{A}\ \wedge\mathcal{E} \qquad \frac{A \wedge B}{B}\ \wedge\mathcal{E}$$

$$\frac{A}{A \vee B}\ \vee\mathcal{I} \qquad \frac{B}{A \vee B}\ \vee\mathcal{I} \qquad \frac{A \vee B \quad \begin{matrix}[A^x] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B^y] \\ \vdots \\ C\end{matrix}}{C}\ \vee\mathcal{E}_{x,y}$$

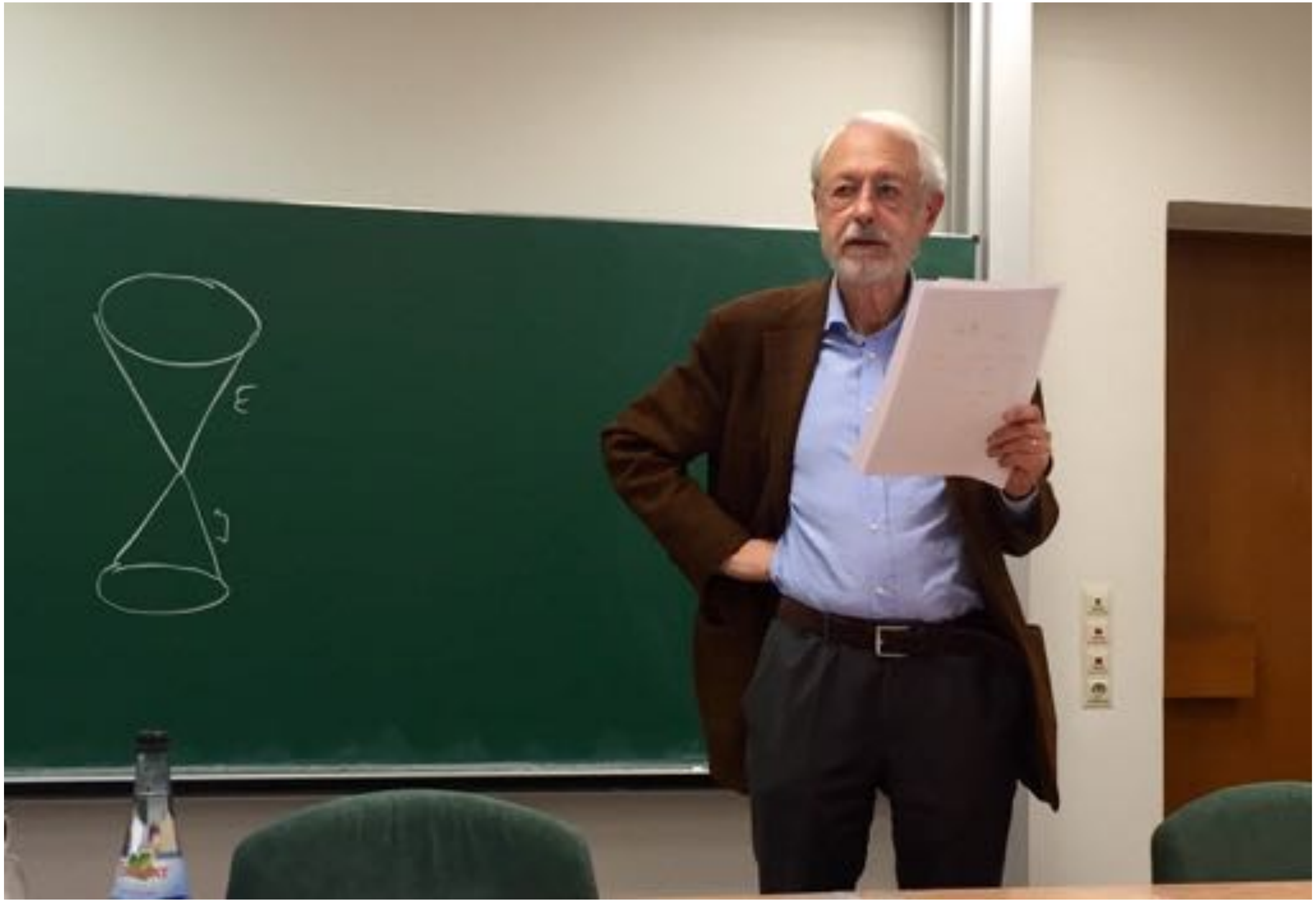
Natural Deduction

Prawitz

Normalization of Natural Deduction ND

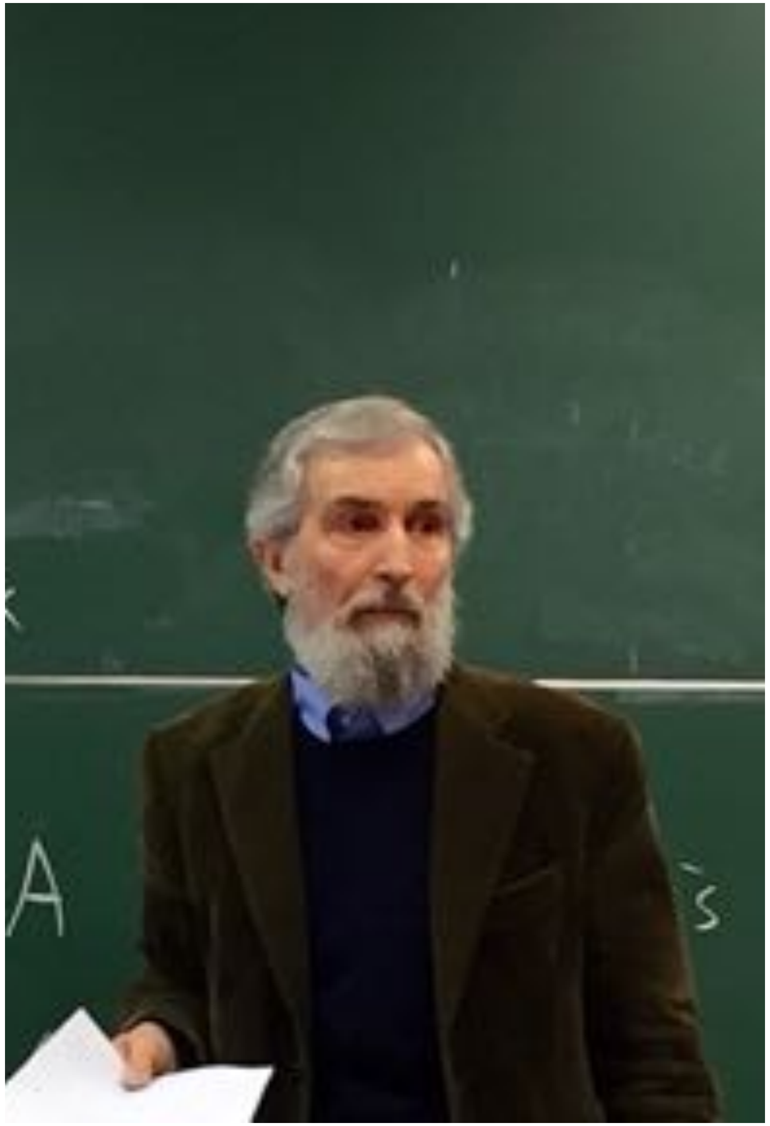
Extensions

Modalities!



Martin-Löf

- Intuitionistic type Theory
- Dependent types
- Universes
- HoTT



Girard

- Normalization of System F
- Linear Logic 1987
- GoI, Ludics, etc…

Logics

Frege: quantifiers

FOL

Hilbert: proofs as mathematical objects

axiom systems

Gentzen: sequents and Natural Deduction

Prawitz: normalization Natural Deduction

Martin-Löf: type theory

Girard: system F

Linear Logic

Modalities

Russel

...there is no one fundamental logical notion of necessity, nor consequently of possibility. If this conclusion is valid, the subject of modality ought to be banished from logic, since propositions are simply true or false...

[Russell, 1905]

Dana Scott

One often hears that modal (or some other) logic is pointless because it can be translated into some simpler language in a first- order way. Take no notice of such arguments. There is no weight to the claim that the original system must therefore be replaced by the new one. What is essential is to single out important concepts and to investigate their properties.

[Advice on Modal Logic, 71]

C.I. Lewis

van Bentham: Modalities

Box A= A is necessarily the case, A holds for all times, A is obligatory,…
Dia A = A is possibly the case, A holds at some time, A is permitted, …

Temporal logic, knowledge operators, BDI models, denotational semantics, effects, security modelling and verification, natural language understanding and inference, databases, etc..

Bisimulation right notion of morphism. Correspondence theory,

Modal logic about structures, not operators

Anil Nerode: Constructive Modalities

Modalities over an Intuitionistic basis:

$\wedge \vee \to \neg$

Constructive modalities ought to be twice as useful?

Usual phenomenon: classical facts can be 'constructivized' in many different ways.

(My work: CS4 with Bierman 1992)

Simpson: Intuitionistic Modalities 1994 thesis

- Operators Box , Diamond (like for all/exists) not inter-definable
- How do these two modalities interact?
- Depends on expected behavior and on tools you want to use.

Solutions add to syntax: hypersequents, labelled deduction systems, (linear) nested sequents, tree-sequents, all add some semantics to syntax (many ways…)

01

Goal: functional programmers, AI scientists, philosophical logicians talking to each other and cooperating

02

Not attained? Communities still largely talking past each other

03

Incremental work on intuitionistic modal logics continues, as well as some of the big research programmes that started it

04

Does it make sense to try to change this status quo?

IMLA: Intuitionistic Modal Logic and Applications

since 1999

IMLAs through Time

- 1999 — First IMLA, FLoC Trento
- 2002 — Second IMLA, FLoC Copenhagen
- 2005 — Third IMLA, LiCS Chicago
- 2008 — Fourth IMLA, LiCS Pittsburgh
- 2011 — Fifth IMLA, CLMPST Nancy
- 2013 — Sixth IMLA, Rio de Janeiro, LU
- 2017 — Seventh IMLA, Toulouse, ESSLLI

I expected 20 years ago…

- Curry-Howard for a BIG collection of intuitionistic modal logics
- Design space for intuitionistic modal logic, for classical logic and how to move from intuitionistic modal to classic modal
- Applications of modal type systems
- Fully worked out dualities for systems
- Off-the-shelf implementations for proof search/proof normalization

Why did I think it would be easy?

- Early successes: systems CS4, Lax, CK
- CS4: On an Intuitionistic Modal Logic (Studia Logica 2000, conference 1992)
  - DIML: Explicit Substitutions for Constructive Necessity (with Neil Ghani and Eike Ritter), ICALP 1998
- Lax Logic: Computational Types from a Logical Perspective (with Benton, Bierman, JFP 1998)
- CK: Basic Constructive Modal Logic. (with Bellin and Ritter, M4M 2001), Kripke semantics for CK (with Mendler 2005), Basic Constructive Modality (with Ritter 2011), Fibrational Modal Type Theory (Ritter 2016).

Modal Axioms

$\Box(A \to B) \to (\Box A \to \Box B)$

$\Box A \to A$

$\Box A \to \Box\Box A$

$\Box(A \to \Diamond B) \to (\Diamond A \to \Diamond B)$

$A \to \Diamond A$

Used by Godel and Girard (for ! only)

Usual intuitionistic axioms plus MP, Nec rules

Constructive S4

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B}\,\Box\mathcal{L} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma, \Delta \vdash \Box A}\,\Box\mathcal{R}$$

$$\frac{\Box\Gamma, A \vdash \Diamond B}{\Delta, \Box\Gamma, \Diamond A \vdash \Diamond B}\,\Diamond\mathcal{L} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}\,\Diamond\mathcal{R}$$

Why it isn't easy…

- Natural Deduction is problematic, as discovered by Wadler and others for LL.
- Issue is PROMOTION rule

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

Substitution!

Any proof using Promotion

$$\frac{\dfrac{\Box A_1 \qquad \Box A_2}{B}}{\Box B}$$

Any Modus Ponens proof finishing in one assumption

$$\frac{C \to \Box A_1 \qquad C}{\Box A_1}$$

$$\frac{\dfrac{C \to \Box A_1 \qquad C}{\Box A_1} \qquad \Box A_2}{B}$$

BUT cannot apply PROMOTION anymore!!

Abramsky's "Computational Interpretation of Linear Logic" (1993),
a calculus that does not satisfy substitution.
There is no substitute for Linear Logic (Wadler)

Category theory to the rescue

Benton, Bierman, de Paiva and Hyland solved the problem for Linear Logic in TLCA 1993. Bierman and de Paiva (Amsterdam 1992, journal 2000) used the same solution for modal logic.

The solution builds in the substitutions into the rule as

$$\frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k \quad \Box A_1, \ldots, \Box A_k \vdash B}{\Gamma \vdash \Box B}(\Box I)$$

Prawitz uses a notion of "essentially modal subformula" to guarantee substitutivity in his monograph. Several 'improvements'.
Two solutions: isn't this enough?

Not nearly enough…

- Logicians want all the modal systems they already have
- Programmers want extensions of modal type theories to deal with their own problems: authentication, flow control, temporal verification, staged computation, abstract syntax, effects, etc.
- HoTT researchers want different kinds of modality, e.g. proof-irrelevance
- Only discussed a propositional basis, FOL/IFOL a can of worms, over dependent type theory, a pandora-box of worms…

Divergent design choices

(even at prop level)

Example (Distribution)

$$\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$$

$$\Diamond\bot \rightarrow \bot$$

Example (Introduction of Box)

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \quad \text{vs.} \quad \frac{\Gamma \quad [xRy] \vdash y : A}{\Gamma \vdash x : \Box A}$$

Prawitz vs. Fitch style ND?

Modal Cubes

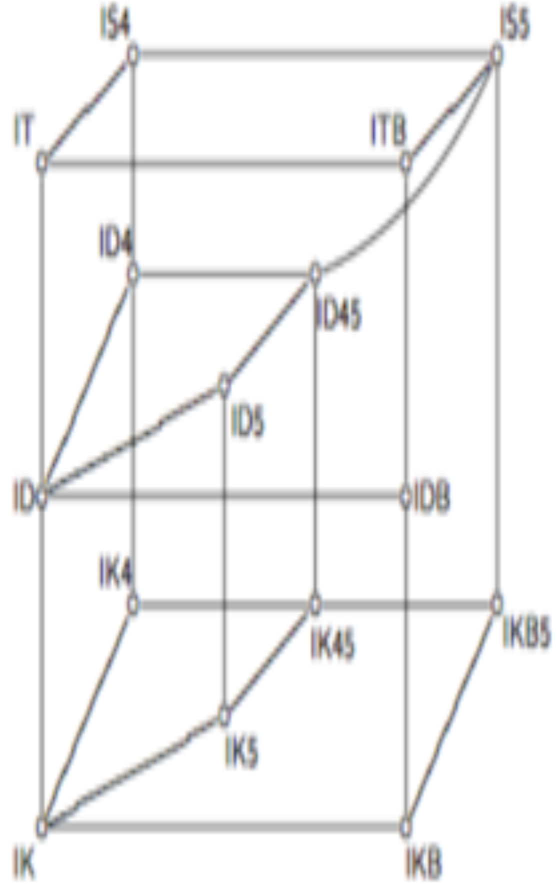
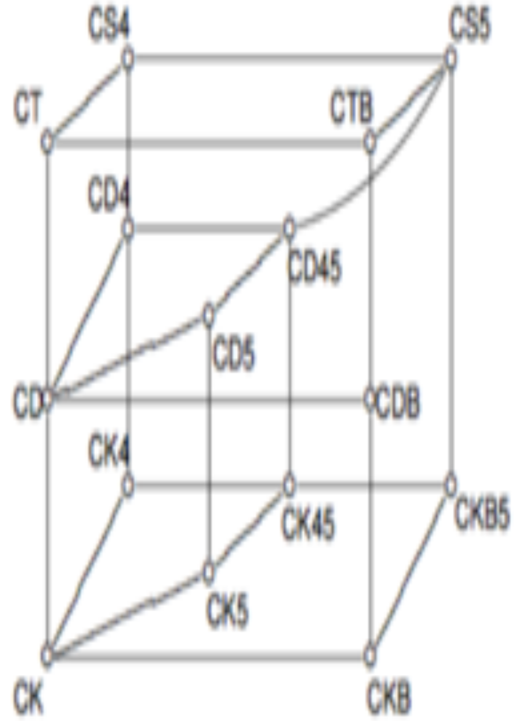


Fig. 2. The intuitionistic "modal cube"



Why it Matters

- Reasoning about [concurrent] programs
  Pnueli, The Temporal Logic of Programs, 1977.
  ACM Turing Award, 1996.
- Reasoning about hardware; model-checking
  Clarke, Emerson, Synthesis of Synchronization Skeletons for Branching Time Temporal Logic, 1981.
  Bryant, Clarke, Emerson, McMillan; ACM Kanellakis Award, 1999
- Knowledge Representation
  From frames to KL-ONE to Description Logics
  MacGregor1987, Baader et al. 2003 **Semantic Web!**

They are also essential in AI, Cognitive Science, Philosophy, Decision Theory, etc...

Thanks Frank Pfenning!

Most applications of formal methods in computing use Modal logic

After this?

- Panorama of Curry-Howard for constructive modal logics
- Plenty of recent work on pure syntax from Bruennler, Strassburger, Pimentel, Lellmann and and many others
- Many applications of the ideas of constructive modal logic
- Many interesting papers on specific applications, see Jagadhesan et al, Jeffrey FRP, etc
- Still lacking an over-arching framework? proposals by Shulman, Licata and others coming from HoTT

Overarching Frameworks?

- Licata, Shulman and Riley (FSCD 2017) Abstract
- "We define a general framework that abstracts the common features of many intuitionistic sub-structural and modal logics / type theories. The framework is a sequent calculus / normal-form type theory parametrized by a *mode theory*, which is used to describe the structure of contexts and the structural properties they obey. [...] Additionally, we give an equivalent semantic presentation of these ideas, in which a mode theory corresponds to a 2-dimensional cartesian multicategory, the framework corresponds to another such multicategory with a functor to the mode theory, and the logical connectives make this into a bifibration."
- Great!
- But "bifibrations of 2-multicategories"?

Prawitz vs. Fitch style ND

- Trees vs cascades (Standefer)
- Borghuis (1994), Martini & Masini (1996)
- [cascade] "systems have not been the focus of meta-theoretic investigation, such as normalization results, as much as Gentzen–Prawitz tree systems."
- Somewhat expected that they would be equivalent under provability for min, int and class logic, but hard to prove. proved by Standefer (2017).
- Unclear what is the relation for modal logics.
- Ranald Clouston (2017) Fitch-style modal lambda calculi apps: guarded recursion, nominal type theory, clocked type theory

Fitch-style (Box) Type Theory

- There is a collection of works now using Fitch-style type theory (Borghius 1998 thesis). We mentioned it in passing in 2001 (Bellin et al)
- The latest installment (Gratzer et al Oct 2021) "Modalities and Parametric Adjoints" says:
- Unfortunately, the development of modal type theories is fraught with difficulties. [...] The admissibility of substitution is a central property of type theory, and indeed of all logic.[...] This might seem like a small technical point, but in practice it is a crucial failing. The problem arises when we try to use a Fitch-style type theory T as an internal language."
- MTT, DRA, MLTTlock, FitchTT based on parametric right adjoints PRAs
- It's complicated!

Too many Fitch TTs?

- "All three approaches have their strengths and weaknesses. The Bierman-de Paiva style of delayed substitutions is conceptually clear, but difficult to use and implement. Moreover, it does not readily adapt to support multiple modalities, at least not when they interact in a nontrivial way. On the other hand, the split-context approach has proven practical whenever the modalities interact in certain convenient ways (see e.g. Shulman [36]). However, this is the exception and not the rule.
- In contrast, the Fitch-style approach is supported by a single universal property which fully determines the modality up to isomorphism—just as with standard connectives, like dependent products and sums. Thus, one might be led to believe that Fitch-style calculi are the preferred formalism. Alas, it is not difficult to see that they suffer from a number of technical disadvantages. We illustrate these using a specific theory, viz. the calculus of dependent right adjoints."
- I need to understand these issues, as it seems to me that our work on explicit substitutions for modal+ expl subs for CIC solves the issues in a cleaner way.

But cleanness, like Beauty is in the eye of the beholder

Parting Thoughts

- The Proof Theory of Modal logic is difficult, Segerberg in Handbook of Logic 1984
- Solutions add to syntax: hypersequents, nested sequents, tree sequents, labelled systems, etc.
- Requiring a proof relevant categorical semantics makes sense
- But doing it over a dependent type theory adds difficulties
- Intuitions come from diff applications: notions of time, cohesion, truncation, proof-irrelevance, globality, nominal type theory, guarded and clocked theory, authentication, provenance, etc...
- But also from AI and philosophical logic, BDI models, public announcements, multiagents, etc...
- Plenty of work to do. Who wants to help?

Thanks!

Special thanks to Agata Ciabattoni, Nicola Olivetti, Elaine Pimentel!