# Dialectica Categories Revisited

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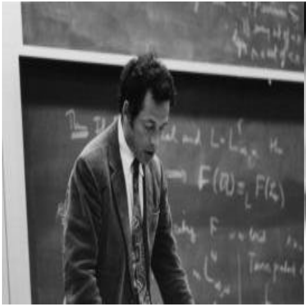
# Thanks!

Johan and Declan for the invitation today
Sol and Grisha for the first invitation!
Lauri for the friendship



# Personal stories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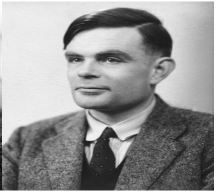

# Personal stories

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# What I want to cover

My very old thesis:

- 4 chapters, 4 main theorems.
- All of them: C is a categorical model of a $L$ logic
- Start from C cartesian closed cat + coproducts + (...)
- Thm 1: DC is a model of !-free ILL
- Thm 2: DC+! (! co-free monoidal comonad) model of ILL
- Thm 3: GC (simple dial cat) a model of (!,?)-free CLL/FILL
- Thm 4: GC+! (comp mon comonad) a model of FILL/CLL, distributive laws

# This Talk

A short version:

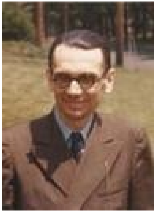
- 2 chapters, 2 main theorems.
- Of the form: C is a categorical model of a $L$ logic
- Start from C cartesian closed cat + coproducts + (...)
- apply Dialectica construction to it $\rightarrow$ DialC
- Thm 1: DialC is a model of !-free ILL
- Thm 2: DialC+! (! co-free monoidal comonad) model of ILL
- Why this is interesting?

# This Talk

A short version:

- 2 chapters, 2 main theorems.
- All of the form: C is a categorical model of a $L$ logic (ND such that..)
- Start from C cartesian closed cat + coproducts + (...)
- apply Dialectica construction to it $\rightarrow$ DialC
- Thm 1: DialC is a model of !-free intuitionistic Linear Logic
- Thm 2: DialC+! (! co-free monoidal comonad) model of ILL
- Why this is interesting?

# Gödel's Dialectica Interpretation



The interpretation is named after the Swiss journal Dialectica where it appeared in a special volume dedicated to Paul Bernays 70th birthday in 1958.

I was originally trying to provide an internal categorical model of the Dialectica Interpretation. The categories I came up with proved to be a model of Linear Logic

# Dialectica (from Wikipedia)

$A_D(u; x)$ quantifier-free formula defined inductively:

$$
\begin{array}{lcl}
(P)_D & \equiv & P \ (P \text{ atomic}) \\
(A \wedge B)_D(u, v; x, y) & \equiv & A_D(u; x) \wedge B_D(v; y) \\
(A \vee B)_D(u, v, z; x, y) & \equiv & (z = 0 \rightarrow A_D(u; x)) \wedge (z \neq 0 \rightarrow B_D(v; y)) \\
(A \rightarrow B)_D(f, F; u, y) & \equiv & A_D(u; Fuy) \rightarrow B_D(fu; y) \\
(\exists z A)_D(u, x; z) & \equiv & A_D(u; x) \\
(\forall z A)_D(f; y, z) & \equiv & A_D(fz; y)
\end{array}
$$

**Theorem (Dialectica Soundness, Gödel 1958)**

*Whenever a formula $A$ is provable in Heyting arithmetic then there exists a sequence of closed terms $t$ such that $A_D(t; y)$ is provable in system $T$. The sequence of terms $t$ and the proof of $A_D(t; y)$ are constructed from the given proof of $A$ in Heyting arithmetic.*

# Dialectica Categories

Gödel's Dialectica: an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type $T$.

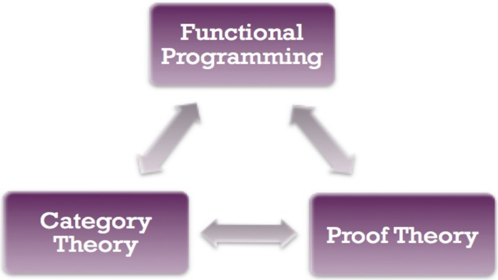
basic idea: translate every formula $A$ of HA to $A^D = \exists u \forall x. A_D$, where $A_D$ is quantifier-free.

Use: If HA proves $A$ then system $T$ proves $A_D(t, y)$ where $y$ is string of variables for functionals of finite type, $t$ a suitable sequence of terms not containing $y$

Goal: to be as constructive *as possible* while being able to interpret all of classical arithmetic (Troelstra)

Philosophical discussion of how much it achieves $\Rightarrow$ another talk

# Categorical Models

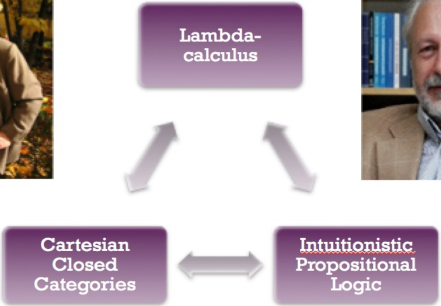



Types are formulae/objects in appropriate category,
Terms/programs are proofs/morphisms in the category,
Logical constructors are 'appropriate' categorical constructions.
Most important: Reduction is proof normalization (**Tait**)
Outcome: Transfer results/tools from logic to CT to CScience

# + Curry-Howard Correspondence

1963

1965



# Linear Logic

A proof theoretic logic described by Jean-Yves Girard in 1986.

Basic idea: assumptions cannot be discarded or duplicated. They must be used exactly once – just like dollar bills

Other approaches to accounting for logical resources before.

Great win of Linear Logic: Account for resources when you want to, otherwise fall back on traditional logic, $A \to B$ iff $!A \multimap B$

## Dialectica Categories

Hyland suggested that to provide a categorical model of the Dialectica Interpretation, one should look at the functionals corresponding to the interpretation of logical implication.

I looked and instead of finding a cartesian closed category, found a monoidal closed one

Thus the categories in my thesis proved to be models of Linear Logic

## Resources in Linear Logic

In Linear Logic formulas denote resources. Resources are premises, assumptions and conclusions, as they are used in logical proofs. For example:

$\$1 \multimap \text{latte}$

If I have a dollar, I can get a Latte

$\$1 \multimap \text{cappuccino}$

If I have a dollar, I can get a Cappuccino

$\$1$

I have a dollar

Using my dollar premise and one of the premisses above, say '$\$1 \multimap$ latte' gives me a latte but the dollar is gone
Usual logic doesn't pay attention to uses of premisses, $A$ implies $B$ and $A$ gives me $B$ but I still have $A$

# Linear Implication and (Multiplicative) Conjunction

Traditional implication: $A, A \to B \vdash B$

$A, A \to B \vdash A \wedge B$ Re-use $A$

Linear implication: $A, A \multimap B \vdash B$

$A, A \multimap B \nvdash A \otimes B$ Cannot re-use $A$

Traditional conjunction: $A \wedge B \vdash A$ Discard $B$

Linear conjunction: $A \otimes B \nvdash A$ Cannot discard $B$

Of course: $!A \vdash !A \otimes !A$ Re-use

$!A \otimes B \vdash I \otimes B \cong B$ Discard

# Challenges of modeling Linear Logic

Traditional categorical modeling of intuitionistic logic

formula $A \rightsquigarrow$ object $A$ of appropriate category

$A \wedge B \rightsquigarrow A \times B$ (real product)

$A \to B \rightsquigarrow B^A$ (set of functions from $A$ to $B$)

These are real products, so we have projections ($A \times B \to A, B$) and diagonals ($A \to A \times A$) which correspond to deletion and duplication of resources

**Not linear!!!**
Need to use *tensor products* and *internal homs* in CT
**Hard**: how to define the “make-everything-usual"operator "!"

# Category *DialC*

Start with a cat $C$ that is cartesian closed (with some other nice properties) Then build a new category $DialC$.

Objects are relations in $C$, triples $(U, X, \alpha)$, $\alpha : U \times X \to 2$, so either $u\alpha x$ or not.

Maps are pairs of maps in $C$. A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of maps in $C$, $(f : U \to V, F : U \times Y \to X)$ such that a 'semi-adjunction condition' is satisfied: for $u \in U, y \in Y$, $u\alpha F(u, y)$ implies $fu\beta y$. (**Note direction and dependence!**)

**Theorem1: (de Paiva 1987) [Linear structure]**

The category *DialC* has a symmetric monoidal closed structure (and products, weak coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

## Can we give some intuition for these objects?

Blass makes the case for thinking of problems in computational complexity. Intuitively an object of *DialC*

$$A = (U, X, \alpha)$$

can be seen as representing a problem.
The elements of $U$ are instances of the problem, while the elements of $X$ are possible answers to the problem instances.
The relation $\alpha$ checks whether the answer is correct for that instance of the problem or not.

(Superpower games?)

## Examples of objects in *DialC*

1. The object $(\mathbb{N}, \mathbb{N}, =)$ where $n$ is related to $m$ iff $n = m$.
2. The object $(\mathbb{N}^{\mathbb{N}}, \mathbb{N}, \alpha)$ where $f$ is $\alpha$-related to $n$ iff $f(n) = n$.
3. The object $(\mathbb{R}, \mathbb{R}, \leq)$ where $r_1$ and $r_2$ are related iff $r_1 \leq r_2$
4. The objects $(2, 2, =)$ and $(2, 2, \neq)$ with usual equality inequality.

# Tensor product in *DialC*

Given objects $(U, X, \alpha)$ and $(V, Y, \beta)$ it is natural to think of $(U \times V, X \times Y, \alpha \times \beta)$ as a tensor product.

This construction does give us a bifunctor

$$\otimes\colon DialC \times DialC \to DialC$$

with a unit $I = (1, 1, id_1)$.

Note that this is not a product.
There are no projections $(U \times V, X \times Y, \alpha \times \beta) \to (U, X, \alpha)$.
Nor do we have a diagonal functor
$\Delta\colon DialC \to DialC \times DialC$, taking
$(U, X, \alpha) \to (U \times U, X \times X, \alpha \times \alpha)$

# Internal-hom in *DialC*

To “internalize"the notion of map between problems, we need to consider the collection of all maps from $U$ to $V$, $V^U$, the collection of all maps from $U \times Y$ to $X$, $X^{U \times Y}$ and we need to make sure that a pair $f\colon U \to V$ and $F\colon U \times Y \to X$ in that set, satisfies the dialectica condition:

$$\forall u \in U, y \in Y, u\alpha F(u, y) \to fu\beta y$$

This give us an object in *DialC* $(V^U \times X^{U \times Y}, U \times Y, \beta^\alpha)$
The relation $\beta^\alpha\colon V^U \times X^{U \times Y} \times (U \times Y) \to 2$ evaluates a pair $(h, H)$ of maps on the pair of elements $(u, y)$ and checks the dialectica implication between the relations.

# Internal-hom in DialC

Given objects $(U, X, \alpha)$ and $(V, Y, \beta)$ we can internalize the notion of morphism of *DialC* as the object $(V^U \times X^{U \times Y}, U \times Y, \beta^\alpha)$

This construction does give us a bifunctor

$$* \multimap *\colon DialC \times DialC \to DialC$$

This bifunctor is contravariant in the first coordinate and covariant in the second, as expected

The kernel of our first main theorem is the adjunction

$$A \otimes B \to C \text{ if and only if } A \to [B \multimap C]$$

where $A = (U, X, \alpha), B = (V, Y, \beta)$ and $C = (W, Z, \gamma)$

# Products and Coproducts in *DialC*

Given objects $(U, X, \alpha)$ and $(V, Y, \beta)$ it is natural to think of $(U \times V, X + Y, \alpha \circ \beta)$ as a categorical product in DC.

Since this is a relation on the set $U \times V \times (X + Y)$, either this relation has a $(x, 0)$ or a $(y, 1)$ element, and hence the $\circ$ symbol only 'picks' the correct relation $\alpha$ or $\beta$.

However, we do not have coproducts. It is only a **weak-coproduct** enough for the logic/type theory

**Theorem: (de Paiva 1987) [Linear structure]**

The category DialC has a symmetric monoidal closed structure (and products, weak coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

# What about the Modality?

We need an operation on objects/propositions such that:

$!A \to !A \otimes !A$ (duplication)

$!A \to I$ (erasing)

$!A \to A$ (dereliction)

$!A \to !!A$ (digging)

Also ! should be a functor, i.e $(f, F) : A \to B$ then $!(f, F) : !A \to !B$

**Theorem: linear and usual logic together**

There is a **monoidal** comonad ! in $DC$ which models exponentials/modalities and recovers Intuitionistic (and Classical) Logic.

Take $!(U, X, \alpha) = (U, X^*, \alpha*)$, where $(-)^*$ is the free commutative monoid in $C$.

# (Cofree) Modality !

To show this works we need to show several propositions:

- ! is a monoidal comonad: there is a natural transformation $m_{(-,-)} : !A \otimes !B \to !(A \otimes B)$ and $m_I : I \to !I$ satisfying many comm diagrams
- ! induces a commutative comonoid structure on $!A$
- $!A$ also has naturally a coalgebra structure induced by the comonad !
- The comonoid and coalgebra structures interact nicely.

There are plenty of other ways to phrase these conditions. The more usual way nowadays seems to be

**Theorem: Linear and non-Linear logic together**

There is a symmetric **monoidal** adjunction between *DialC* and its **cofree** coKleisli category for the monoidal comonad ! above.

# Cofree Modality !

Old way: “There is a monoidal comonad ! on a linear category *DialC* satisfying (lots of conditions)"and

**Theorem: Linear and non-Linear logic together**

The coKleisli category associated with the comonad ! on *DialC* is cartesian closed.

To show cartesian closedness we need to show:

$$Hom_{Kl!}(A\&B, C) \cong Hom_{Kl!}(A, [B, C]_{Kl!})$$

The proof is then a series of equivalences that were proved before:
$Hom_{Kl!}(A\&B, C) \cong Hom_{DialC}(!(A\&B), C) \cong$
$Hom_{DialC}(!A \otimes !B, C) \cong Hom_{DialC}(!A, [!B, C]_{DialC}) \cong$
$Hom_{kl!}(A, [!B, C]_{DialC}) \cong Hom_{kl!}(A, [B, C]_{kl!})$
(Seely, 1989; de Paiva, 1989)

# What is the point of (these) Dialectica categories?

First, the construction provides a model of Linear Logic, instead of constructive logic. This allows us to see where the assumptions in Gödel's argument (hacks?) are used (new work with Trotta and Spadetto showing where G needs IP, MP and skolemization)

It justifies linear logic in terms of a more traditional logic tool and conversely explains the more traditional work in terms of a 'modern' (linear, resource conscious) decomposition of concerns.

Dialectica categories provide models of linear logic as well as an internal representation of the dialectica interpretation. Modeling the exponential ! is hard, first model to do it. Still (one of) the best ones.

# What is the point of (these) Dialectica categories?

30 years on: maybe too much emphasis on LL earlier

Now fashion is polynomials, clear connection to Dialectica categories (Moss 2022)

Dialectica construction can be applied to many more logical systems (modal dialectica, monotone, ...). Categorical constructions have only scratched the surface.

(At least) Three PhDs on Dialectica Models of Dependent Type Theory: Biering 2007, von Glehn 2016, Moss 2018.

Connections to many other areas: lenses in FP, partial compilers (Plotkin), games with bidding (Hedges2014), etc...

# Applications

Applications to Set Theory (S. G da Silva since 2013)

ACT adjoint school 2020 – Petri nets for Chemistry and Biology

Relationship with POLY and Lenses: AMS-MRC this May

Multiagent systems based on LL and dialectica spaces

Game models (Winskel 2022, Koenig 2021)

Relationship with automated differentiation/neural nets?

automata theory (Pradic 2020)

- https://github.com/vcvpaiva/DialecticaCategories

# Conclusions

Original Dialectica category
with its symmetric cofree comonad is back in fashion

Category DialC is a cat model of L.

Curry-Howard correspondence works (several logical systems L)

Categorical Proof Theory is expanding! (higher-order cats)

Later: Dialectica Petri nets (di Lavore, Leal),
Dialectica-Kolmogorov problems (G da Silva),
Dialectica abstract machine (Pedrot),
Dialectica automata (Pradic),
Dialectica domain theory games (Winskel)

# Thank you!

# Some References

(see https://github.com/vcvpaiva/DialecticaCategories)

A.Blass, *Questions and Answers: A Category Arising in Linear Logic, Complexity Theory, and Set Theory*, Advances in Linear Logic (ed. J.-Y. Girard, Y. Lafont, and L. Regnier) London Math. Soc. Lecture Notes 222 (1995).

de Paiva, *A dialectica-like model of linear logic*, Category Theory and Computer Science, Springer, (1989) 341–356.

de Paiva, *The Dialectica Categories*, In Proc of Categories in Computer Science and Logic, Boulder, CO, 1987. Contemporary Mathematics, vol 92, American Mathematical Society, 1989 (eds. J. Gray and A. Scedrov)

P. Hofstra, *The Dialectica Monad and its cousins*, Proc in honour of M. Makkai, 2011.