Dialectica and Godel Doctrines

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(Joint work with Davide Trotta and Matteo Spadetto)

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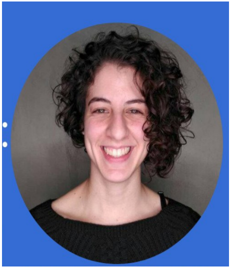
Thanks!



Many thanks to our organizers!
Ana Luiza Tenorio stands for the large group, some 12 people are on the website, but possibly many others on the ground.

Dialectica Interpretation

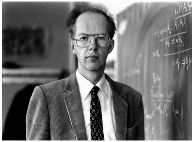
Dialectica Interpretation (Gödel 1958): an interpretation of intuitionistic arithmetic HA in a quantifier-free theory of functionals of finite type **System $\mathbf{T}$**.

Idea: translate every formula A of HA to

$$A^D = \exists u \forall x A_D$$

where A_D is quantifier-free.

Dialectica Interpretation



Application (Gödel 1958): if HA proves A, then System T proves $A_D(t, x)$, where x is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing x).

Goal: to be as **constructive** as possible, while being able to interpret all of classical Peano arithmetic (Troelstra).

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes.*, Dialectica, 12(3-4):280–287. (Translation in Gödel's Collected Works)

Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(A \to B)^D$:

$$(A \to B)^D = \exists V, X. \forall u, y. (A_D(u, X(u, y)) \to B_D(V(u), y)).$$

Intuition: Given a witness u in U for the hypothesis A_D, there exists a function V assigning a witness $V(u)$ to B_D. Moreover, from a counterexample y to the conclusion B_D, we should be able to find a counterexample $X(u, y)$ for the hypothesis A_D.

Feferman et al editors (1986), *Kurt Gödel: Collected Works: Volume II*, Oxford University Press.

Dialectica interpretation

Troelstra (p 226 Collected Works Gödel) from Spector (1962)

$$[\exists u \forall x. A_D(u,x) \to \exists v \forall y. B_D(v,y)] \leftrightarrow^{(i)}$$

$$[\forall u(\forall x A_D(u,x) \to \exists v. \forall y(B_D(v,y))] \leftrightarrow^{(ii)}$$

$$[\forall u \exists v(\forall x. A_D(u,x) \to \forall y B_D(v,y))] \leftrightarrow^{(iii)}$$

$$[\forall u \exists v \forall y(\forall x A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(iv)}$$

$$[\forall u \exists v \forall y \exists x(A_D(u,x) \to B_D(v,y))] \leftrightarrow^{(v)}$$

$$\exists V, X \forall u, y(A_D(u, X(u,y)) \to B_D(V(u), y))$$

where (i) and (iii) are intuitionistic, but (ii) requires **Independence of Premise**, (iv) requires **Markov Principle** and (v) requires two uses of the **axiom of choice**.

Dialectica interpretation

Hence translation involves three logical, non-intuitionistic, principles:

1. **Principle of Independence of Premise** (IP)

$$(A \to \exists v.B(v)) \to \exists v.(A \to B(v))$$

2. a generalisation/modification of **Markov Principle** (MMP)

$$(\forall x.A(x) \to B(y)) \to \exists x.(A(x) \to B(y))$$

3. the **axiom of choice** (AC)

$$\forall y.\exists x.A(x,y) \to \exists V.\forall y.A(V(y),y)$$

Categorical Dialectica Construction

Dialectica category (de Paiva 1988): Given a category C with finite limits, one can build a new category $\mathfrak{Dial}(\mathsf{C})$, whose objects have the form $A = (U, X, \alpha)$ where α is a subobject of $U \times X$ in C; ***think*** of this object as representing the formula

$$\exists u \forall x \alpha(u, x).$$

A map from $\exists u \forall x \alpha(u, x)$ to $\exists v \forall y \beta(v, y)$ can be thought of as a pair (f_0, f_1) of terms/maps, subject to the entailment condition

$$\alpha(u, f_1(u, y)) \vdash \beta(f_0(u), y).$$

(First internalisation of the Dialectica interpretation!)

Categorical Dialectica Constructions

Most of the work in the original Dialectica categories (de Paiva 1989, 1991) was on the categorical structure needed to model Linear Logic (Girard 1987).

We described symmetric monoidal closed categories with appropriate (co)monads, modelling the modality !

This model is pretty cool! Lots of recent work on it, 30 years later. BUT:

Generalization: initial construction has been generalized for arbitrary fibrations, by Hyland, Biering, Hofstra, von Glehn, Moss, etc.

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de Paiva (1991), *The Dialectica categories*, Cambridge PhD Thesis.
Hofstra (2011), *The dialectica monad and its cousins.*, A tribute to M. Makkai
Trotta, Spadetto and de Paiva (2021), *The Gödel fibration.*, MFCS 2021
Trotta, Spadetto and de Paiva (2022), *Gödel Doctrines.*, LFCS 2022

Dialectica via Doctrines

Recently Trotta, Spadetto and V. describe a categorical version of Dialectica in terms of (Lawvere's) **doctrines**.

In three arxiv preprints, two conference papers and two journal papers their work explains how modelling of the dialectica interpretation using doctrines (or fibrations) is very tight.

But **why** do we do it?

Isn't the modelling using categories enough?

Trotta, Spadetto and de Paiva (2023), *Dialectica principles via Gödel doctrines*, TCS
Trotta, Spadetto and de Paiva (2022), *Dialectica logical principles: not only rules.*, JLC 2022

Dialectica via Doctrines

Two reasons:

- First-order is of course more expressive than propositional logic, sometimes we need the extra expressivity;
- Much tighter correspondence between the logic and the category theory, as exemplified by the Dialectica logical principles paper

In particular we get the ability to show why the internalisation of morphisms work.

Dialectica via Doctrines

How well does the construction of the Dialectica categories (or doctrines) capture the essential ingredients of Gödel's original interpretation?

1. Given a doctrine P, when is there a doctrine P' such that $\mathfrak{Dial}(P') \cong P$?
2. When such doctrine P' exists, how do we find it?

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   Such a P' exists precisely when P is a **Gödel doctrine**
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2. When such doctrine P' exists, how do we find it? P' is given by the **quantifier-free elements** of the Gödel doctrine P

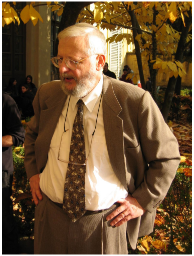
Dialectica via Doctrines

The Dialectica translation requires some classical principles:
independence of premise(IP)
Markov principle (MP)
and the axiom of choice (AC).

How can we see these principles in our categorical modelling?

Can these categories and these principles be described in more conceptual terms, for example, in terms of universal properties?

Doctrines



Lawvere defined hyperdoctrines, we start with less.

Definition

A **doctrine** is just a functor from a category $\mathcal{C}$ with finite products, to Pos, the category of posets.

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$$

Doctrines

Example: Let C be a category with finite limits. The doctrine

$$Sub_{\mathsf{C}} \colon C^{\mathrm{op}} \longrightarrow \mathsf{Pos}$$

assigns to an object A in C the poset $Sub_{\mathsf{C}}(A)$ of subobjects of A in C, that is (equivalence classes of) monics $M \to A$.

For an arrow $B \xrightarrow{f} A$, $Sub_C(A) \xrightarrow{Sub_C(f)} Sub_C(B)$ is given by pulling a subobject back along f.

Existential and Universal Doctrines

Definition (existential/universal doctrines)

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is ***existential*** (resp. ***universal***) if, for every A_1 and A_2 in $\mathcal{C}$ and every projection $A_1 \times A_2 \xrightarrow{\pi_i} A_i$, $i = 1, 2$, the functor:

$$PA_i \xrightarrow{P_{\pi_i}} P(A_1 \times A_2)$$

has a left adjoint $\exists_{\pi_i}$ (resp. a right adjoint $\forall_{\pi_i}$), and these satisfy the ***Beck-Chevalley conditions***.

(Trotta (TAC 2020): The existential completion exists and satisfies all 2-categorical properties you may want. Ditto for the universal completion.)

Doctrines and quantifier-free formulas

We want a suitable universal property to represent predicates that are quantifier-free, categorically.

We will have dual definitions for existential and universal quantifiers.

The paper defines:

- existential splitting predicates,
- existential-free predicates,
- doctrines P with enough existential-free predicates.

(as well as their duals)

Trotta, Spadetto and de Paiva (2023), *Dialectica principles via Gödel doctrines*, TCS

Definition (existential-free predicates)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Pos}$ be an existential doctrine and let I be an object of $\mathcal{C}$. We say the predicate $\alpha(i)$ of the fibre $P(I)$ is ***existential-free*** if for every arrow $A \to I$ of $\mathcal{C}$ such that $\alpha(f(a) \vdash (\exists b : B)\beta(a, b)$ in $P(A)$, where $\beta(a, b)$ is a predicate in $P(A \times B)$, there exists a unique arrow $g : A \to B$ such that $\alpha(f(a)) \vdash \beta(a, g(a))$.

Similarly, we can define universal-free predicates of universal doctrines.

Cf. *Dialectica Logical Principles via Doctrines*, arXiv 2205.0709.

Definition (Gödel doctrine)

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is called a ***Gödel doctrine*** if:

1. the category $\mathcal{C}$ is cartesian closed;
2. the doctrine P is existential and universal;
3. the doctrine P has enough existential-free predicates;
4. the existential-free objects of P are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_{\pi}(\alpha)$ is existential-free for every projection π from A;
5. the sub-doctrine $P'\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ of the existential-free predicates of P has enough universal-free predicates.

a mouthful! without 5. we call it a ***Skolem doctrine***.

Definition (Dialectica doctrine)

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathrm{Pos}$ be a doctrine whose base category C is cartesian closed. The **dialectica doctrine** $\mathfrak{Dial}(P)\colon \mathcal{C}^{op} \longrightarrow \mathrm{Pos}$ is defined as the functor sending an object I into the poset $\mathfrak{Dial}(P)(I)$ defined as follows:

objects are quadruples (I, U, X, α) where I, X and U are objects of the base category $\mathcal{C}$ and $\alpha \in P(I \times U \times X)$;

partial order: we say that $(I, U, X, \alpha) \leq (I, V, Y, \beta)$ if there exists a pair (f_0, f_1), where $I \times U \xrightarrow{f_0} V$ and $I \times U \times Y \xrightarrow{f_1} X$ are morphisms of $\mathcal{C}$ such that:

$$\alpha(i, u, f_1(i, u, y)) \leq \beta(i, f_0(i, u), y).$$

This is a direct adaptation to the proof irrelevant setting of Hofstra's definition.

Theorem (Hofstra 2011)

If $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is a doctrine, then there is an isomorphism $\mathfrak{Dial}(P) \cong (P^{\forall})^{\exists}$ which is natural in P.

(Here $Q^{\forall}$ and $Q^{\exists}$ denote the universal and the existential completions of any doctrine Q.)

Theorem

Every Gödel doctrine P is equivalent to the Dialectica completion $\mathfrak{Dial}(P')$ of the full subdoctrine P' of P consisting of the quantifier-free predicates of P.

Theorem (1. Gödel doctrine objects)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ *be a Gödel doctrine and* α *be an element of* $P(A)$. *Then there exists a quantifier-free predicate* α_D *of* $P(I \times U \times X)$ *such that:*

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U.\forall x : X.\alpha_D(i, u, x).$$

This theorem shows that Gödel doctrines allow us to describe their quantifier-free objects. In a Gödel doctrine every formula admits a presentation of the precise form used in the Dialectica translation.

Theorem (2. Gödel doctrine maps)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ *be a Gödel doctrine. Then for every* $A_D \in P(I \times U \times X)$ *and* $B_D \in P(I \times V \times Y)$ *quantifier-free predicates of* P *we have that:*

$$i : I \mid \exists u.\forall x.A_D(i,u,x) \vdash \exists v.\forall y.B_D(i,v,y)$$

if and only if there exists $I \times U \xrightarrow{f_0} V$ *and* $I \times U \times Y \xrightarrow{f_1} X$ *such that:*

$$u : U, y : Y, i : I \mid A_D(i,u,f_1(i,u,y)) \vdash B_D(i,f_0(i,u),y).$$

This theorem shows that morphisms of the dialectica categories correspond to implication in the Gödel doctrine.

Theorem (3. Skolemization principle)

Every Gödel doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathrm{Pos}$ *validates the* ***Skolemisation principle****, that is:*

$$i : I \mid \forall u.\exists x.\alpha(i,u,x) \dashv\vdash \exists f.\forall u.\alpha(i,u,fu)$$

where $f : X^U$ *and* fu *denote the evaluation of* f *on* u*, whenever* $\alpha(i,u,x)$ *is a predicate in* $I \times U \times X$*.*

This theorem shows that the skolemisation required by the Dialectica is modelled in Gödel doctrines. But we need more than implication for dialectica.

Gödel hyperdoctrines

A ***hyperdoctrine*** is a functor from a cartesian closed category to the category Hey of Heyting algebras

$$P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$$

satisfying: for every arrow $A \xrightarrow{f} B$ in $\mathcal{C}$, the homomorphism of Heyting algebras $P_f \colon P(B) \longrightarrow P(A)$ has a left adjoint $\exists_f$ and a right adjoint $\forall_f$ satisfying the Beck-Chevalley conditions.

Definition (Gödel hyperdoctrine)

A hyperdoctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ is called a ***Gödel hyperdoctrine*** when P is a Gödel doctrine.

Theorem (5. Independence of Premise)

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ *satisfies the **Rule of Independence of Premise**: whenever* β *in* $P(A \times B)$ *and* α *in* $P(A)$ *is an existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash \alpha(a) \to \exists b.\beta(a, b)$$

implies that

$$a : A \mid \top \vdash \exists b.(\alpha(a) \to \beta(a, b)).$$

Theorem (6. Modified Markov Rule)

Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ *satisfies the following* ***Modified Markov Rule****: whenever* $\beta_D \in P(A)$ *is a quantifier-free predicate and* $\alpha \in P(A \times B)$ *is an existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash (\forall b.\alpha(a, b)) \to \beta_D(a)$$

implies that

$$a : A \mid \top \vdash \exists b.(\alpha(a, b) \to \beta_D(a)).$$

Summarizing

Used existential and universal doctrines (and their completions) to provide notions of quantifier-free formulae

Showed that the Gödel doctrines satisfy:
Dialectica Normal Form
Soundness of Implication
Skolemisation
Independence of Premise
Markov Principle

Obtained a very faithful categorical description of the Dialectica interpretation.

Future work?

AMS-MRC Applied Category Theory 2022
4 working subgroups:

- polynomials (Niu)
- fibrational optics (Capucci)
- games (Koenig)
- Petri processes (Aten)

Other work:

- more computability? more realizability?
- more fibrations? more type theory?
- more logic? Hilbert ε-operators?
- more programming? more differentiation?
- more concurrency? Winskel games?

Conclusions

Several categorical models of Gödel's Dialectica

Extended and generalized the original models.

Try to make clear the connections to realizability tripos and toposes and many others

Thank you!

Elegant mathematics will of itself tell a tale, and one with the merit of simplicity. This may carry philosophical weight. But that cannot be guaranteed: in the end one cannot escape the need to form a judgement of significance."J.M.E. Hyland, 2004

Some References

- de Paiva, *The Dialectica Categories*, In Proc of Categories in Computer Science and Logic, Boulder, CO, 1987.
- D. Trotta, M. Spadetto, V. de Paiva, *The Gödel Fibration*, MFCS 2021.
- D. Trotta, M. Spadetto, V. de Paiva, *Dialectica Logical Principles*, LFCS 2022.
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- D. Trotta, M. Spadetto, V. de Paiva, *Dialectica Principles via Gödel Doctrines*. TCS 2023

Hofstra's Dialectica tripos

Dialectica Tripos. We show that the dialectica tripos can also be incorporated. For a description of this tripos we refer to [1].

The dialectica tripos has a generic object

$$\Sigma = \{(X, Y, A) | X, Y \subseteq \mathbb{N}, A \subseteq X \times Y, 0 \in A \cap Y\}$$

and the preorder in the fibre over 1 is given by

$$\begin{aligned} (X, Y, A) \vdash (X', Y', A') \quad \Leftrightarrow \quad \exists f, F \in \mathbb{N} : \quad & f \in (X \Rightarrow X'), \\ & F \in (X \times Y' \Rightarrow Y), \\ & A(x, F(x, y)) \text{ implies } A'(fx, y) \end{aligned}$$

and in the fibre over M we require this uniformly in all $m \in M$. We order the generic element by putting

$$(X, Y, A) \leq (X', Y', A') \Leftrightarrow X \subseteq X', Y' \subseteq Y, A \subseteq A'.$$