Dialectica Petri Nets

Valeria de Paiva
Joint work with
Elena di Lavore and Wilmer Leal
InterCats 2022

Mar 2022

Today is International Women's Day!

Gender Pay Gap Bot
@PayGapApp

Employers, if you tweet about International Women's Day, I'll retweet your gender pay gap 👀 #IWD2022 #BreakTheBias

gender-pay-gap.service.gov.uk Joined March 2021

38 Following **43.5K** Followers

Followed by On This Day She, joao, and 8 others you follow

Intercats

Jules Hedges

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Lenses for philosophers

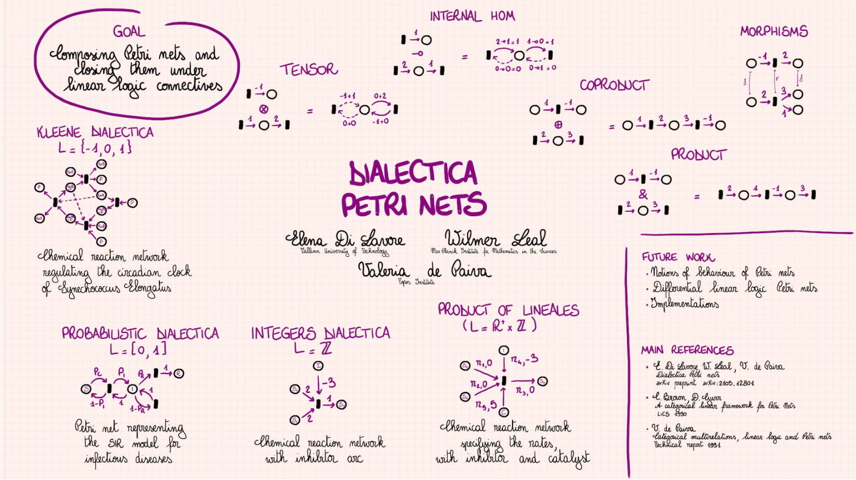
Posted on Aug 16, 2018 by juleshedges

Lens tutorials are the new monad tutorials, I hear. (This is neat, since monads and lenses were both discovered in the year 1958.) The thing is, after independently rediscovering lenses and working on them for a year and a half before Jeremy Gibbons made the connection, I have a very different perspective on them. This post is based on a talk I gave at the 7th international workshop on bidirectional transformations in Nice. My aim is to move fast and break things, where the things in question are your preconceptions about what lenses are and what they can be used for. Much of this will be a history of lenses, which includes at least 9 independent rediscoveries.

The earliest discovery of lenses (that I know of) was by Kurt Gödel in 1958, as part of his dialectica interpretation. (Actually it presented at a lecture at Yale in 1941,

Many Thanks Elena and Wilmer!

Also Xiaoyan Li, Jade Master, Eigil Rischel, and all the organizers of the ACT School!



DIALECTICA PETRI NETS

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GOAL

Composing Petri nets and closing them under linear logic connectives

KLEENE DIALECTICA
$L = \{-1, 0, 1\}$

Chemical reaction network regulating the circadian clock of Synechococcus Elongatus

PROBABILISTIC DIALECTICA
$L = [0, 1]$

Petri net representing the SIR model for infectious diseases

INTEGERS DIALECTICA
$L = \mathbb{Z}$

Chemical reaction network with inhibitor arc

PRODUCT OF LINEALES
$(L = \mathbb{R}^+ \times \mathbb{Z})$

Chemical reaction network specifying the rates, with inhibitor and catalyst

TENSOR

INTERNAL HOM

COPRODUCT

PRODUCT

MORPHISMS

FUTURE WORK

- Notions of behaviour of Petri nets
- Differential linear logic Petri nets
- Implementations

MAIN REFERENCES

- E. Di Lavore, W. Leal, V. de Paiva
  Dialectica Petri nets
  arXiv preprint arXiv:2105.12801
- C. Brown, D. Gurr
  A categorical linear framework for Petri Nets
  LiCS 1990
- V. de Paiva
  Categorical multirelations, linear logic and Petri nets
  Technical report 1991

An Accidental Concurrency Theorist

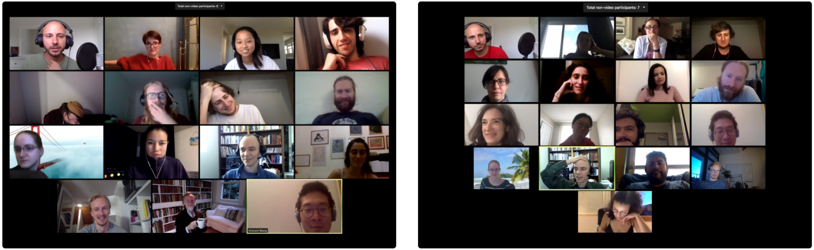
Carolyn Brown and Douglas Gurr

An Accidental Concurrency Theorist

Old work:

- Valeria de Paiva. The Dialectica Categories. Phd Thesis, Cambridge, 1988.
- Carolyn Brown, Doug Gurr. A Categorical Linear Framework for Petri Nets. LICS 1990: 208-218
- Carolyn Brown. Linear logic and Petri nets: Categories, algebra and proof, Phd Thesis, Edinburgh, 1990.
- Valeria De Paiva. Categorical Multirelations, Linear Logic and Petri Nets, Technical Report, 1991.
- Carolyn Brown, Doug Gurr, and Valeria de Paiva. A Linear Specification Language for Petri Nets (Aarhus Technical Report), 1991.

Thirty years later, did you say?

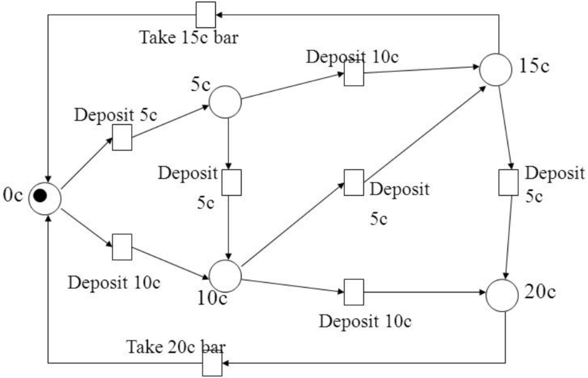


- ACT School 2020: Dialectica categories of Petri nets with Jade Master, Elena di Lavore, Xiaoyan Li, Wilmer Leal and Eigil Rischel
- Poster at ACT2021, arXiv 2105.12801, submitted 2022
- phd theses: Brown 1990, Lilius 1991, Gupta 1994, Farwer 1999, Stehr 2002, Misra 2004.

Outline

- Petri nets
- Dialectica construction
- Constructions on Petri nets
- Different arcs

Simple example: a vending machine



Brown/Gurr original idea

- squares are events (in E), circles are conditions (in B), tokens move through the net.
- pre and post conditions on events are relations like the ones in the Dialectica construction
- need to stay with *safe* nets, as relations either hold (1) or don't (0), no multiplicity

Why Petri Nets?

- Modeling is hard. People seem to like Petri nets.
- Concurrency is hard. People seem to like Petri nets (non-determinism vs causal independence).
- Huge number of types of nets: colored, stochastic, higher-order, etc..
- Huge number of papers/books/systems
- Natural application of Linear Logic
- Recent work of Baez/Master and independently of Lopes/Hauesler/Benevides (dynamic logic)

What's the problem?

> *A long-standing problem in the theory of Petri nets has been the lack of any clear methodology for producing a compositional theory of nets (and indeed the lack of even a clear notion as to what a map between nets should be).*
>
> *Brown and Gurr, 1990*

Winskel suggests that Petri nets and other models of concurrency can be profitably cast into an **algebraic** framework.
Many people followed this suggestion with several categorical models of Petri nets proposed.
Our proposal is to use Dialectica morphisms, so let us check the Dialectica construction for Linear Logic

Dialectica Categories

Definition

The dialectica category Dial$_2$(**Sets**) has as objects triples $A = (U, X, \alpha)$, where U, X are sets and α is an ordinary relation between U and X, $\alpha : U \times X \to 2$. (so either u and x are α related, $\alpha(u, x) = 1$ or not, $\alpha(u, x) = 0$.)

A map from $A = (U, X, \alpha)$ to $B = (V, Y, \beta)$ is a pair of functions (f, F), $f\colon U \to V$ and $F\colon Y \to X$ such that

$$\begin{tikzcd}
U \arrow[d, "f"'] & X \arrow[l, "\alpha"'] \arrow[dl, phantom, "\Downarrow"] \\
V & Y \arrow[l, "\beta"'] \arrow[u, "F"']
\end{tikzcd}$$

$$\forall u \in U, \forall y \in Y \quad \alpha(u, Fy) \text{ implies } \beta(fu, y)$$

Dialectica construction variations

Sending pairs of sets, U, X plus α to $\mathrm{Dial}_2(\mathbf{Sets})$.

- can change **Sets** in $\mathrm{Dial}_2(\mathbf{Sets})$, to a generic cartesian closed category $\mathbf{C}$
- can change a subset of a product $\alpha \subseteq U \times X$ to a map $\alpha\colon U \times X \to 2$
- can change truth-values 2 to a **lineale** L making $\mathrm{Dial}_L(\mathbf{Sets})$, see technical report
- can change maps to make it more like Chu-spaces, equality
- can change objects by adding more relations

A lineale L is simply the poset version of a symmetric monoidal closed category. Named for analogy with *quantales*
Could say L is a monoidal closed poset or a residuated commutative lattice or an exponential monoid.

Dialectica Petri net

Definition

The category of safe Dialectica Petri nets has as objects quadruples $A = (E, B, pre, post)$, where E, B are sets of events and conditions, respectively, and $pre, post$ are the pre and post-condition relations between E and B, $pre, post : E \times B \to 2$ that determine the network.

A map from $A = (E, B, pre, post)$ to $B = (E', B', pre', post')$ is a pair of functions (f, F) where $f \colon E \to E'$ and $F \colon B' \to B$ such that

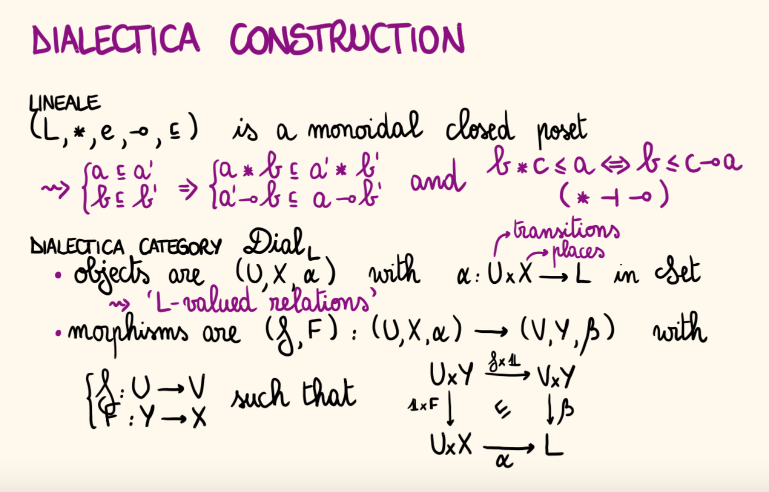
for all e in E, b' in B'

$$pre(e, F(b')) \leq pre(f(e), b') \text{ and } post(e, F(b')) \leq post(f(e), b')$$

(tweaked Brown and Gurr 1990)

More categories of Petri nets

Petri nets have multiplicities in the arcs, e.g.



DIALECTICA CONSTRUCTION

LINEALE

$(L, *, e, \multimap, \leq)$ is a monoidal closed poset

$\rightsquigarrow \begin{cases} a \leq a' \\ b \leq b' \end{cases} \Rightarrow \begin{cases} a * b \leq a' * b' \\ a' \multimap b \leq a \multimap b' \end{cases}$ and $b * c \leq a \Leftrightarrow b \leq c \multimap a$
$(* \dashv \multimap)$

DIALECTICA CATEGORY Dial_L

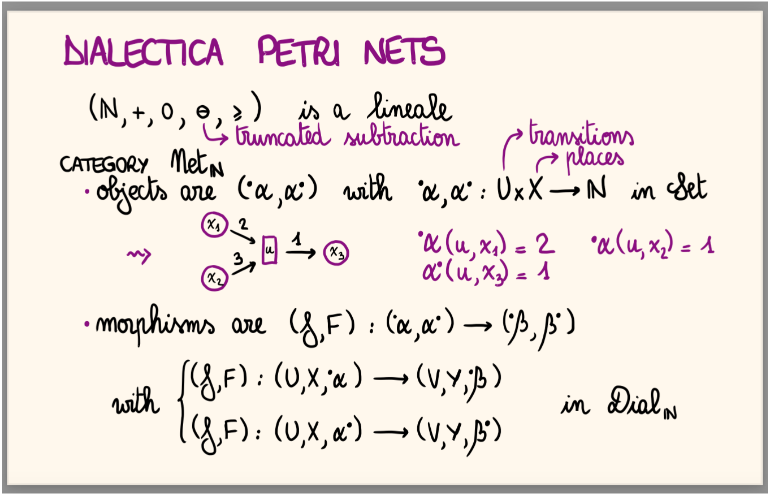
- objects are (U, X, α) with $\alpha : \overset{\text{transitions}}{U} \times \overset{\text{places}}{X} \longrightarrow L$ in Set
  $\rightsquigarrow$ 'L-valued relations'
- morphisms are $(f, F) : (U, X, \alpha) \longrightarrow (V, Y, \beta)$ with

$\begin{cases} f : U \to V \\ F : Y \to X \end{cases}$ such that

$$\begin{tikzcd} U \times Y \arrow[r, "f \times 1"] \arrow[d, "1 \times F"'] \arrow[dr, phantom, "\leq"] & V \times Y \arrow[d, "\beta"] \\ U \times X \arrow[r, "\alpha"'] & L \end{tikzcd}$$

More categories of Petri nets

Write ${}^\circ\alpha, \alpha^\circ$ for *pre*, *post*.



DIALECTICA PETRI NETS

$(\mathbb{N}, +, 0, \ominus, \geq)$ is a lineale

↳ truncated subtraction

CATEGORY $\mathrm{Net}_{\mathbb{N}}$

- objects are $({}^\circ\alpha, \alpha^\circ)$ with ${}^\circ\alpha, \alpha^\circ : U \times X \longrightarrow \mathbb{N}$ in Set

↗ transitions (U)

↗ places (X)

⇝

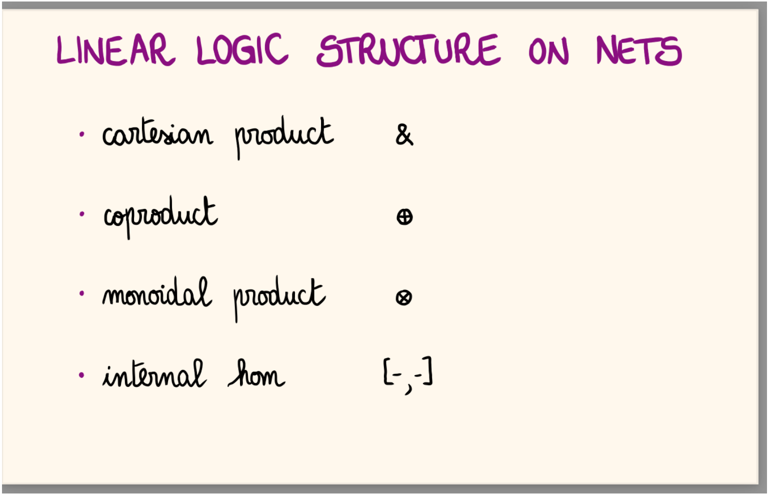
${}^\circ\alpha(u,x_1) = 2$ ${}^\circ\alpha(u,x_2) = 1$

$\alpha^\circ(u,x_3) = 1$

- morphisms are $(f,F) : ({}^\circ\alpha, \alpha^\circ) \longrightarrow ({}^\circ\beta, \beta^\circ)$

with $\begin{cases} (f,F) : (U,X,{}^\circ\alpha) \longrightarrow (V,Y,{}^\circ\beta) \\ (f,F) : (U,X,\alpha^\circ) \longrightarrow (V,Y,\beta^\circ) \end{cases}$ in $\mathrm{Dial}_{\mathbb{N}}$

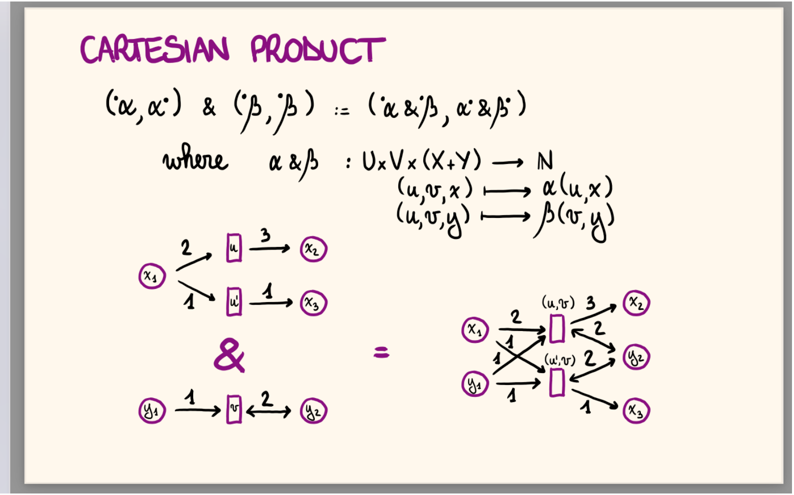
Constructions on nets



LINEAR LOGIC STRUCTURE ON NETS

- cartesian product $\&$
- coproduct $\oplus$
- monoidal product $\otimes$
- internal hom $[-,-]$

Constructions on nets



CARTESIAN PRODUCT

$$({}^\bullet\alpha, \alpha^\bullet) \,\&\, ({}^\bullet\beta, \beta^\bullet) := ({}^\bullet\alpha \,\&\, {}^\bullet\beta, \alpha^\bullet \,\&\, \beta^\bullet)$$

where $\alpha \& \beta : U \times V \times (X+Y) \longrightarrow \mathbb{N}$

$$(u,v,x) \longmapsto \alpha(u,x)$$
$$(u,v,y) \longmapsto \beta(v,y)$$

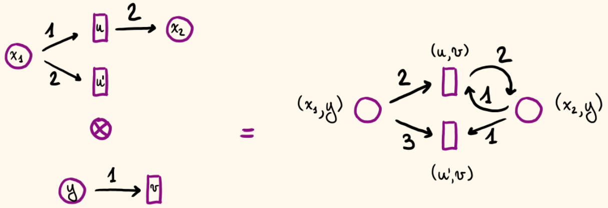
Constructions on nets

MONOIDAL PRODUCT

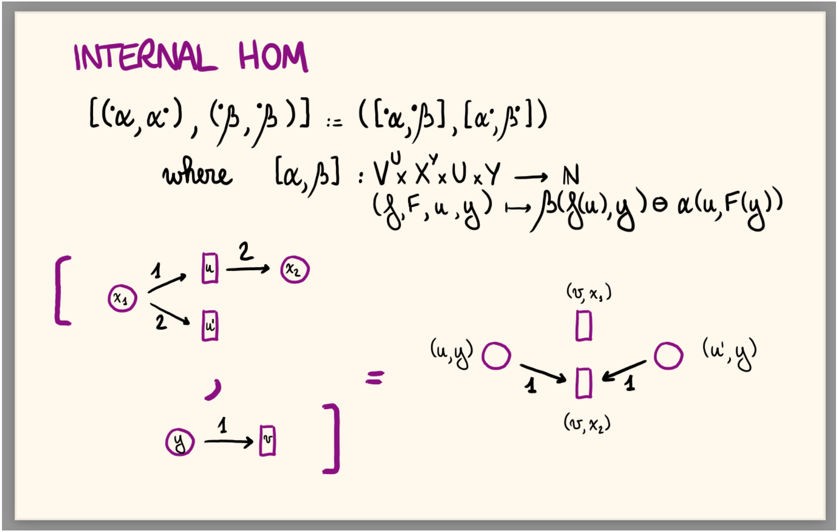
$$({}^{\cdot}\alpha, \alpha^{\cdot}) \otimes ({}^{\cdot}\beta, \beta^{\cdot}) := ({}^{\cdot}\alpha \otimes {}^{\cdot}\beta, \alpha^{\cdot} \otimes \beta^{\cdot})$$

where $\alpha \otimes \beta : U \times V \times X^V \times Y^U \longrightarrow \mathbb{N}$

$(u, v, f, g) \mapsto \alpha(u, f(v)) + \beta(v, g(u))$



Constructions on nets



INTERNAL HOM

$$[({}^\bullet\alpha,\alpha^\bullet),({}^\bullet\beta,{}^\bullet\beta)] := ([{}^\bullet\alpha,{}^\bullet\beta],[\alpha^\bullet,\beta^\bullet])$$

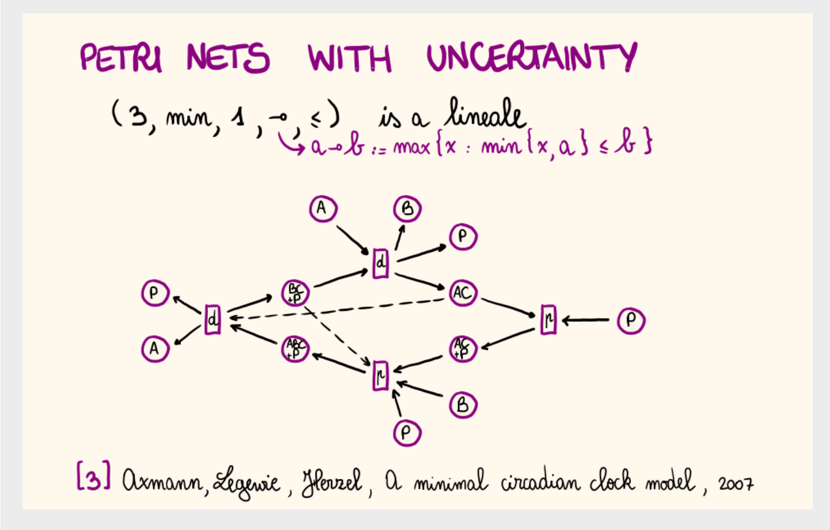
where $[\alpha,\beta] : V^U \times X^Y \times U \times Y \longrightarrow \mathbb{N}$

$(f,F,u,y) \longmapsto \beta(f(u),y) \ominus \alpha(u,F(y))$

Examples of lineales

- $L = 3$ uncertain arcs
- $L = [0, 1]$ arcs with probabilities
- $L = \mathbb{R}^+$ arcs with rates
- $L = \mathbb{Z}$ inhibitor arcs
- $L = L_1 \times L_2$ product of lineales

Changing lineales



PETRI NETS WITH UNCERTAINTY

$(3, \min, 1, \multimap, \leq)$ is a lineale

$a \multimap b := \max\{x : \min\{x, a\} \leq b\}$

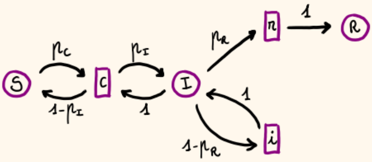
[3] Axmann, Legewie, Herzel, A minimal circadian clock model, 2007

Changing lineales

PROBABILISTIC PETRI NETS

$([0,1], \cdot, 1, \multimap, \leq)$ is a lineale

$$a \multimap b := \begin{cases} b/a & a \geq b \wedge a \neq 0 \\ 1 & a < b \vee a = 0 \end{cases}$$

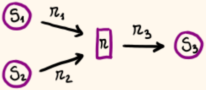


Changing lineales

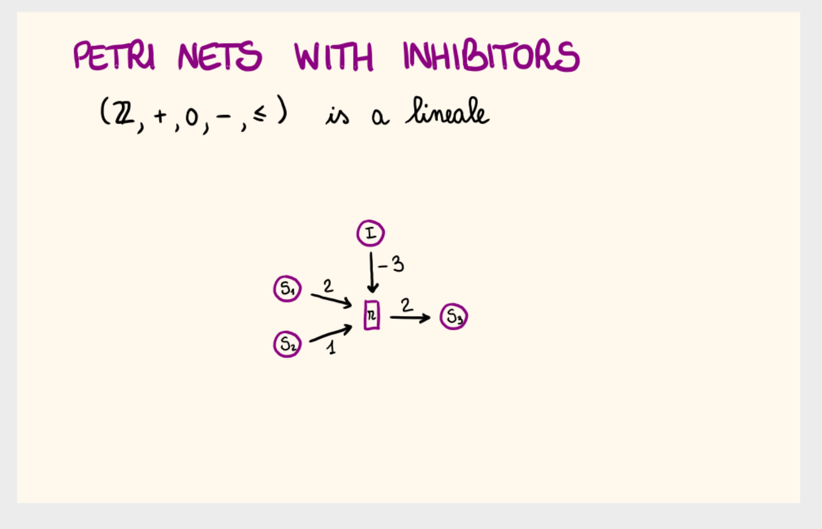
PETRI NETS WITH RATES

$(\mathbb{R}^+, +, 0, \ominus, \geqslant)$ is a lineale

↳ truncated subtraction



Changing lineales



PETRI NETS WITH INHIBITORS

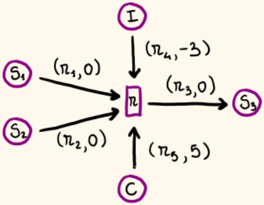
$(\mathbb{Z}, +, 0, -, \leq)$ is a lineale

Changing lineales

PRODUCT OF LINEALES

$(L_1, *_1, e_1, \multimap_1, \leq_1)$ and $(L_2, *_2, e_2, \multimap_2, \leq_2)$ lineales

$\Rightarrow (L_1 \times L_2, *, (e_1, e_2), \multimap, \leq)$ is a lineale



$$\rightsquigarrow \begin{cases} L_1 = \mathbb{R}^+ \\ L_2 = \mathbb{Z} \end{cases}$$

Summing up

- Main gain: different kinds of arc labelling for the same
- compositional theory from beginning
- linear logic connectives can be modelled, use LL language