Dialectica Categorical Constructions

Valeria de Paiva

JMM 2023

Jan, 2023

Thanks!

Applied Category Theory MRC, Dialectica crew, especially Charlotte Aten and Jeremie Koenig!

Dialectica Interpretation

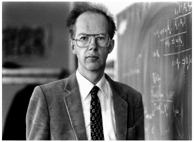
Dialectica Interpretation (Gödel 1958): an interpretation of intuitionistic arithmetic (Heyting arithmetic) HA in a quantifier-free theory of functionals of finite type **System $\mathbf{T}$**.

Idea: translate every formula A of HA to

$$A^D = \exists u \forall x A_D$$

where A_D is quantifier-free.

Dialectica Interpretation



Application: if HA proves A, then System T proves $A_D(t, x)$, where x is a string of variables for functionals of finite type, and t a suitable sequence of terms (not containing x).

Goal: to be as **constructive** as possible, while being able to interpret all of classical Peano arithmetic (Troelstra).

Gödel (1958), *Über eine bisher noch nicht benützte erweiterung des finiten standpunktes.*, Dialectica, 12(3-4):280–287. (Translation in Gödel's Collected Works)

Dialectica interpretation

The most complicated clause of the translation is the definition of the translation of the **implication connective** $(A \to B)^D$:

$$(A \to B)^D = \exists V, X.\forall u, y.(A_D(u, X(u, y)) \to B_D(V(u), y)).$$

Intuition: Given a witness u in U for the hypothesis A_D, there exists a function V assigning a witness $V(u)$ to B_D. Moreover, from a counterexample y to the conclusion B_D, we should be able to find a counterexample $X(u, y)$ for the hypothesis A_D.

Dialectica interpretation

The translation involves three logical principles:

1. **Principle of Independence of Premise** (IP)
2. a generalization of **Markov Principle** (MP)
3. the **axiom of choice** (AC)

Categorical Dialectica Constructions

Dialectica category (de Paiva 1988): Given a category C with finite limits, one can build a new category $\mathfrak{Dial}(\mathsf{C})$, whose objects have the form $A = (U, X, \alpha)$ where α is a subobject of $U \times X$ in C; think of this object as representing the formula

$$\exists u \forall x \alpha(u, x).$$

A map from $\exists u \forall x \alpha(u, x)$ to $\exists v \forall y \beta(v, y)$ can be thought of as a pair (f_0, f_1) of terms/maps, subject to the entailment condition

$$\alpha(u, f_1(u, y)) \vdash \beta(f_0(u), y).$$

(First internalization of the Dialectica interpretation!)

Categorical Dialectica Constructions

Most of the work in the original Dialectica categories was on the categorical structure needed to model Linear Logic (Girard 1987).

We described symmetric monoidal closed categories with appropriate comonads, modelling the modality !

Generalization: the initial construction has been generalized for arbitrary fibrations, by Hyland, Biering, Hofstra, von Glehn, Moss, etc.

---

de Paiva (1988), *The Dialectica categories*, Cambridge PhD Thesis.

Hofstra (2011), *The dialectica monad and its cousins.*, Models, logics, and higher dimensional categories: A tribute to M. Makkai, 2011

Trotta, Spadetto and de Paiva (2021), *The Gödel fibration.*, MFCS 2021

Trotta, Spadetto and de Paiva (2022), *Gödel Doctrines.*, LFCS 2022

Dialectica via Doctrines

Trotta, Spadetto and V. describe a categorical version of Dialectica in terms of (Lawvere's) **doctrines**

How does the construction of the Dialectica categories (or fibrations) capture the essential ingredients of Gödel's original translation?

1. Given a doctrine P, when is there a doctrine P' such that $\mathfrak{Dial}(P') \cong P$?

2. When such doctrine P' exists, how do we find it?

Dialectica via Doctrines

Trotta, Spadetto and V. describe a categorical version of Dialectica in terms of (Lawvere's) **doctrines**

How does the construction of the Dialectica categories (or fibrations) capture the essential ingredients of Gödel's original translation?

1. Given a doctrine P, when is there a doctrine P' such that $\mathfrak{Dial}(P') \cong P$?
   Such a P' exists precisely when P is a **Gödel doctrine**

2. When such doctrine P' exists, how do we find it?

Dialectica via Doctrines

Trotta, Spadetto and V. describe a categorical version of Dialectica in terms of (Lawvere's) **doctrines**

How does the construction of the Dialectica categories (or fibrations) capture the essential ingredients of Gödel's original translation?

1. Given a doctrine P, when is there a doctrine P' such that $\mathfrak{Dial}(P') \cong P$?
   Such a P' exists precisely when P is a **Gödel doctrine**

2. When such doctrine P' exists, how do we find it? P' is given by the **quantifier-free elements** of the Gödel doctrine P

Dialectica via Doctrines

The Dialectica translation requires some classical principles:
independence of premise(IP)
Markov's principle (MP)
and the axiom of choice (AC)

How can we see these principles in the categorical modelling?

Can these categories and these principles be described in more conceptual terms, for example, in terms of universal properties?

Doctrines

Lawvere defined hyperdoctrines, we start with less structure, simply a doctrine.

Definition (doctrine)

A **doctrine** is just a functor from a category $\mathcal{C}$ with finite products, to Pos, the category of posets

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$$

Quantifier Doctrines

Definition (existential/universal doctrines)

A doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is ***existential*** (resp. ***universal***) if, for every A_1 and A_2 in $\mathcal{C}$ and every projection $A_1 \times A_2 \xrightarrow{\pi_i} A_i$, $i = 1, 2$, the functor:

$$PA_i \xrightarrow{P_{\pi_i}} P(A_1 \times A_2)$$

has a left adjoint $\exists_{\pi_i}$ (resp. a right adjoint $\forall_{\pi_i}$), and these satisfy the ***Beck-Chevalley conditions***.

Definition (existential-free predicates)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ be an existential doctrine and let I be an object of $\mathcal{C}$. We say the predicate $\alpha(i)$ of the fibre $P(I)$ is ***existential-free*** if for every arrow $A \to I$ of $\mathcal{C}$ such that $\alpha(f(a) \vdash (\exists b : B)\beta(a,b)$ in $P(A)$, where $\beta(a,b)$ is a predicate in $P(A \times B)$, there exists a unique arrow $g : A \to B$ such that $\alpha(f(a)) \vdash \beta(a, g(a))$.

Similarly, we can define universal-free predicates of universal doctrines.

Cf. *Dialectica Logical Principles via Doctrines*, arXiv 2205.0709.

Definition (Gödel doctrine)

A doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ is called a ***Gödel doctrine*** if:

- the category $\mathcal{C}$ is cartesian closed;
- the doctrine P is existential and universal;
- the doctrine P has enough existential-free predicates;
- the existential-free objects of P are stable under universal quantification, i.e. if $\alpha \in P(A)$ is existential-free, then $\forall_\pi(\alpha)$ is existential-free for every projection π from A;
- the sub-doctrine $P' \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ of the existential-free predicates of P has enough universal-free predicates.

Theorem (1. Gödel doctrine objects)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ *be a Gödel doctrine and* α *be an element of* $P(A)$*. Then there exists a quantifier-free predicate* α_D *of* $P(I \times U \times X)$ *such that:*

$$i : I \mid \alpha(i) \dashv\vdash \exists u : U.\forall x : X.\alpha_D(i,u,x).$$

This theorem shows that Gödel doctrines allow us to describe their quantifier-free objects.

Theorem (2. Gödel doctrine maps)

Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ *be a Gödel doctrine. Then for every* $A_D \in P(I \times U \times X)$ *and* $B_D \in P(I \times V \times Y)$ *quantifier-free predicates of* P *we have that:*

$$i : I \mid \exists u.\forall x.A_D(i,u,x) \vdash \exists v.\forall y.B_D(i,v,y)$$

if and only if there exists $I \times U \xrightarrow{f_0} V$ *and* $I \times U \times Y \xrightarrow{f_1} X$ *such that:*

$$u : U, y : Y, i : I \mid A_D(i,u,f_1(i,u,y)) \vdash B_D(i,f_0(i,u),y).$$

Theorem (3. Skolemization principle)

Every Gödel doctrine $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Pos}$ *validates the* ***Skolemisation principle****, that is:*

$$i : I \mid \forall u.\exists x.\alpha(i,u,x) \dashv\vdash \exists f.\forall u.\alpha(i,u,fu)$$

where $f : X^U$ *and* fu *denotes the evaluation of* f *on* u*, whenever* $\alpha(i,u,x)$ *is a predicate in the context* $I \times U \times X$*.*

Theorem (4. Dial completion)

Every Gödel doctrine P *is equivalent to the Dialectica completion* $\mathfrak{Dial}(P')$ *of the full subdoctrine* P' *of* P *consisting of the quantifier-free predicates of* P*.*

Gödel hyperdoctrines

A ***hyperdoctrine*** is a functor from a cartesian closed category to the category Hey of Heyting algebras

$$P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$$

satisfying: for every arrow $A \xrightarrow{f} B$ in $\mathcal{C}$, the homomorphism of Heyting algebras $P_f\colon P(B) \longrightarrow P(A)$ has a left adjoint $\exists_f$ and a right adjoint $\forall_f$ satisfying the Beck-Chevalley conditions.

Definition (Gödel hyperdoctrine)

A hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ is called a ***Gödel hyperdoctrine*** when P is a Gödel doctrine.

Theorem (5. Independence of Premise)

*Every Gödel hyperdoctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Hey}$ satisfies the **Rule of Independence of Premise**: whenever β in $P(A \times B)$ and α in $P(A)$ is an existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash \alpha(a) \to \exists b.\beta(a, b)$$

implies that

$$a : A \mid \top \vdash \exists b.(\alpha(a) \to \beta(a, b)).$$

Theorem (6. Modified Markov Rule)

Every Gödel hyperdoctrine $P : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Hey}$ *satisfies the following* ***Modified Markov Rule****: whenever* $\beta_D \in P(A)$ *is a quantifier-free predicate and* $\alpha \in P(A \times B)$ *is an existential-free predicate, it is the case that:*

$$a : A \mid \top \vdash (\forall b.\alpha(a, b)) \to \beta_D(a)$$

implies that

$$a : A \mid \top \vdash \exists b.(\alpha(a, b) \to \beta_D(a)).$$

Connecting Theorem

If we assume that for a **Gödel hyperdoctrine** P the existential-free elements are closed under finite conjunctions and implications, then it is the case that:

Theorem

The doctrine P models the Principle of Independence of Premise: whenever β is in $P(A \times B)$ and α in $P(A)$ is an existential-free predicate;
and the Markov Principle: whenever β in $P(A)$ is a quantifier-free predicate and $\alpha \in P(A \times B)$ is an existential-free predicate.

Summarizing:

Showed how to model quantifier-free formulae using Dialectica based doctrines

Proved that the Gödel doctrines satisfy:
Dialectica Normal Form: $\exists u : U \forall x : X . A(u, x)$
Soundness of Implication: $(A \to B)^D \cong A^D \to B^D$
Skolemisation
Independence of Premise
Markov Principle

A very faithful description of the Dialectica interpretation.

Cf. *Dialectica Logical Principles via Doctrines* arXiv 2205.07093.pdf

Biering PhD (2007) Triposes & Toposes provides a picture:

$$
\begin{tikzcd}
 & \mathsf{dn}_m \arrow[rr, shift left, "!"] \arrow[d, shift left, "q_*"] & & \mathsf{d}_m \arrow[ll, shift left, "id"] \arrow[rr, shift left, "F"] \arrow[dll, shift left, "p_*"] \arrow[d, shift left, "H_*"] & & \mathbf{Set} \arrow[ll, shift left, "\nabla"] \\
 & \mathsf{m} \arrow[u, shift left, "q^*"] \arrow[urr, shift left, "p^*"] \arrow[rr, shift left, "v_*"] \arrow[dl, shift left, "\Gamma"] & & \mathsf{e} \arrow[u, shift left, "H^*"] \arrow[ll, shift left, "v^*"] \arrow[rr, shift left, "i_*"] \arrow[dl, shift left, "\Gamma"] \arrow[dr, shift left, "j^*"] & & \mathsf{dn} \arrow[ll, shift left, "i^*"] \\
\mathbf{Set} \arrow[ur, shift left, "\Delta"] \arrow[rr, "\cong"] & & \mathbf{Set} \arrow[ur, shift left, "\Delta"] & & \mathsf{d} \arrow[ul, shift left, "j_*"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathbf{DN}_m \arrow[d, "q"] & & \mathbf{Dia}_m \arrow[d, "H"] & & \\
 & \mathbf{Mod} \arrow[rr, "v"] & & \mathbf{Eff} \arrow[rr, hook, "i"] \arrow[dr, hook, "j"] & & \mathbf{DN} \\
\mathbf{Mod}_{\neg\neg} \arrow[ur, hook] \arrow[rr, "\cong"] & & \mathbf{Eff}_{\neg\neg} \arrow[ur, hook] & & \mathbf{Dia} &
\end{tikzcd}
$$

Conclusions

Several categorical models of the Dialectica interpretation exist now

They extend and generalize the dialectica categories original models in phd thesis

Now want to try and make clearer the connections to realizability tripos and toposes

Thank you!

Some References

- D. Trotta, M. Spadetto, V. de Paiva, *The Gödel Fibration*, MFCS 2021. arXiv 2104.14021.
- D. Trotta, M. Spadetto, V. de Paiva, *Dialectica Logical Principles*, LFCS 2022. arXiv 2109.08064
- D. Trotta, M. Spadetto, V. de Paiva, *Dialectica Principles via Gödel Doctrines.*. arXiv 2205.0709 (submitted)

extra slide: Spector-Troelstra

$$
\begin{array}{rcl}
(A \Rightarrow B)^D & = & (\exists u \forall x A_D \Rightarrow \exists v \forall y B_D)^D \\
& \equiv & [\, \forall u (\forall x A_D \Rightarrow \exists v \forall y B_D)\,]^D \\
& \equiv & [\, \forall u \exists v (\forall x A_D \Rightarrow \forall y B_D)\,]^D \\
& \equiv & [\, \forall u \exists v \forall y (\forall x A_D \Rightarrow B_D)\,]^D \\
& \equiv & [\, \forall u \exists v \forall y \exists x (A_D \Rightarrow B_D)\,]^D \\
& \equiv & \exists \mathbf{V} \mathbf{X} \forall u y [A_D(u, \mathbf{X}(u,y)) \Rightarrow B_D(
\end{array}
$$

extra slide: Hofstra's Dialectica tripos

Dialectica Tripos. We show that the dialectica tripos can also be incorporated. For a description of this tripos we refer to [1].

The dialectica tripos has a generic object

$$\Sigma = \{(X, Y, A) | X, Y \subseteq \mathbb{N}, A \subseteq X \times Y, 0 \in A \cap Y\}$$

and the preorder in the fibre over 1 is given by

$$\begin{aligned}(X, Y, A) \vdash (X', Y', A') \quad \Leftrightarrow \quad \exists f, F \in \mathbb{N} : \quad & f \in (X \Rightarrow X'),\\ & F \in (X \times Y' \Rightarrow Y),\\ & A(x, F(x, y)) \text{ implies } A'(fx, y)\end{aligned}$$

and in the fibre over M we require this uniformly in all $m \in M$. We order the generic element by putting

$$(X, Y, A) \leq (X', Y', A') \Leftrightarrow X \subseteq X', Y' \subseteq Y, A \subseteq A'.$$