Modal Type Theory

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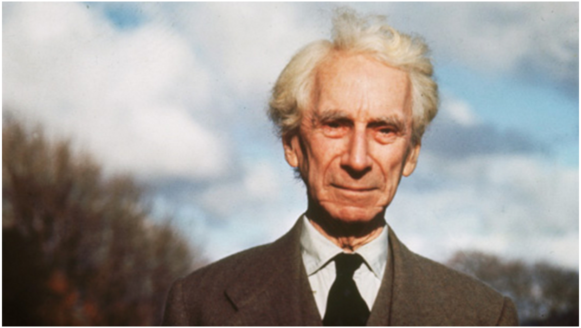
Modal Logic



...there is no one fundamental logical notion of necessity, nor consequently of possibility. If this conclusion is valid, the subject of modality ought to be banished from logic, since propositions are simply true or false...

[Russell, 1905]

Modal Logic

One often hears that modal (or some other) logic is pointless because it can be translated into some simpler language in a first-order way. Take no notice of such arguments. There is no weight to the claim that the original system must therefore be replaced by the new one. What is essential is to single out important concepts and to investigate their properties. [Scott, 1971]

Siding with Scott

Today:
Constructive Modal Logics: Challenges and opportunities An “easy” dependent system

Siding with Scott

Intuitionistic Modal Logic and Applications (IMLA)
is a loose association of researchers, meetings
and a certain amount of mathematical common ground.
IMLA stems from the hope that
philosophers,
mathematical logicians
and computer scientists
would share information and tools when investigating
intuitionistic modal logics and modal type theories,
if they knew of each other's work.

Intuitionistic Modal Logics and Applications IMLA

Six Workshops and five special volumes:

- M. Fairtlough, M. Mendler, Eugenio Moggi (eds.) Modalities in Type Theory, Mathematical Structures in Computer Science, (2001)
- V. de Paiva, R. Goré, M. Mendler (eds.), Modalities in constructive logics and type theories, Journal of Logic and Computation, (2004)
- V. de Paiva, B. Pientka (eds.) Intuitionistic Modal Logic and Applications (IMLA 2008), Inf. Comput. 209(12): 1435-1436 (2011)
- V. de Paiva, M. Benevides, V. Nigam and E. Pimentel (eds.), Proceedings of the 6th Workshop on Intuitionistic Modal Logic and Applications (IMLA 2013), Electronic Notes in Theoretical Computer Science, Volume 300, (2014)
- N. Alechina, V. de Paiva (eds.) Intuitionistic Modal Logics (IMLA2011), Journal of Logic and Computation (2015)

Intuitionistic Modal Logic

Basic idea:
Modalities over an Intuitionistic basis

Research questions:

- which modalities?
- which intuitionistic basis?
- why? how? which proof systems?
- why so many?
- how to choose?
- can relate to others?
- which are the important theorems?
- which are the most useful applications?

Modal Logic

- Modalities: the most successful logical framework in CS
- Temporal logic, knowledge operators, BDI models, denotational semantics, effects, security modelling and verification, natural language understanding and inference, databases, etc..
- Logic used both to create logical representation of information and to reason about it
- But usually only **classical** modalities...

Classical Modalities in Computer Science

- Reasoning about [concurrent] programs
  Pnueli, The Temporal Logic of Programs, 1977.
  ACM Turing Award, 1996.
- Reasoning about hardware; model-checking
  Clarke, Emerson, Synthesis of Synchronization Skeletons for Branching Time Temporal Logic, 1981.
  Bryant, Clarke, Emerson, McMillan; ACM Kanellakis Award, 1999
- Knowledge representation
  From frames to KL-ONE to Description Logics
  MacGregor87, Baader et al03

Thanks for the slide Frank Pfenning!

Intuitionistic Modal Logic

Basic idea: Modalities over an Intuitionistic Basis

- which modalities?
- which intuitionistic basis?
- **why? how?** my take, based on Curry-Howard correspondence...
- why so many?
- how to choose?
- can relate to others?
- which are the important theorems?
- which are the most useful applications?

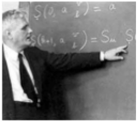
Constructive reasoning in CS

- What: Reasoning principles that are safer
- if I ask you whether "is there an x such that P(x)?",
- I'm happier with an answer "yes, x_0", than with an answer "yes, for all x it is not the case that not $P(x)$".
- Why: want reasoning to be as precise and safe as possible
- How: constructive reasoning as much as possible, classical if need be, but tell me where...

Constructive Logic

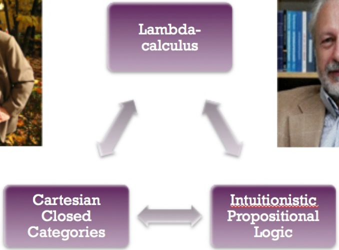
- a logical basis for programming via Curry-Howard correspondences
- short digression...
- Modalities useful in CS
- Examples from applications abound (Monadic Language, Separation Logic, DKAL, etc..)
- Constructive modalities ought to be twice as useful?
- But which constructive modalities?
- Usual phenomenon: classical facts can be 'constructivized' in many different ways. Hence constructive notions multiply

+ Curry-Howard Correspondence



1963

1965



Trivial Case of Curry Howard

Add λ terms to Natural Deduction:

$$\frac{\Gamma, x\colon A \vdash t\colon B}{\Gamma \vdash \lambda x\colon A.t\colon A \to B}\ (\to I)$$

$$\frac{\Gamma \vdash t\colon A \to B \quad \Gamma \vdash u\colon A}{\Gamma \vdash tu\colon B}\ (\to E)$$

Works well for conjunction, disjunction too (not so pretty, the "shame of ND").

Constructive Modal Logics

- Operators Box, Diamond (like forall/exists), not interdefinable
- How do these two modalities interact?
- Depends on expected behavior and on tools you want/can accept
- Collection of articles on why is the proof theory of modal logic difficult
- child poster of difficulty: S5
- Adding to syntax: hypersequents, labelled deduction systems, adding semantics to syntax (many ways...)

Schools of Constructive Modal logic

- Control of Hybrid Systems, Nerode et al, from 1990
- Logic of Proofs, Justification Logics, Artemov, from 1995
- Judgemental Modal Logic, Pfenning et al, from 2001
- Separation Logic, Reynolds and O'Hearn
- Modalities as Monads, Moggi et al, Lax Logic, Mendler et al, from 1990,
- Simpson framework, Negri sequent calculus
- Avron hyper-sequents, Dosen's higher-order sequents, Belnap display calculus, Bruennler/Strassburger, Poggiolesi and others "Nested sequents"

What's the state of play?

- IMLA's goal: functional programmers talking to philosophical logicians and vice-versa
- Not attained, so far
- Communities still largely talking past each other
- Incremental work on intuitionistic modal logics continues, as well as some of the research programmes above
- Does it make sense to try to change this?

What did I expect fifteen years ago?

- Fully worked out Curry-Howard for a collection of intuitionistic modal logics
- Fully worked out design space for intuitionistic modal logic, for classical logic and how to move from intuitionistic modal to classic modal
- Full range of applications of modal type systems
- Fully worked out dualities for desirable systems
- Collections of implementations for proof search/proof normalization

Why did I think it would be easy?

Some early successes. Systems: CS4, Lax, CK

- CS4: On an Intuitionistic Modal Logic (with Bierman, Studia Logica 2000, conference 1992)
  - DIML: Explicit Substitutions for Constructive Necessity (with Neil Ghani and Eike Ritter), ICALP 1998
- Lax Logic: Computational Types from a Logical Perspective (with Benton, Bierman, JFP 1998)
- CK: Basic Constructive Modal Logic. (with Bellin and Ritter, M4M 2001), Kripke semantics for CK (with Mendler 2005),

Constructive S4 (CS4)

- This is the better behaved modal system, used by Gödel and Girard
- CS4 motivation is category theory, because of proofs, not simply provability
- Usual intuitionistic axioms plus MP, Nec rule and

Modal Axioms

$\Box(A \to B) \to (\Box A \to \Box B)$
$\Box A \to A$
$\Box A \to \Box\Box A$
$\Box(A \to \Diamond B) \to (\Diamond A \to \Diamond B)$
$A \to \Diamond A$

CS4 Sequent Calculus

S4 modal sequent rules already discussed in 1957 by Ohnishi and Matsumoto:

$$\frac{\Gamma, A \vdash B}{\Gamma, \Box A \vdash B} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

$$\frac{\Box\Gamma, A \vdash B}{\Box\Gamma, \Diamond A \vdash B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}$$

Cut-elimination works, for classical and intuitionistic basis.

CS4 Natural Deduction Calculus

But ND was more complicated.
The rule

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

(called by Wadler *promotion* in Linear Logic, where $\Box = !$) led to some controversy.
As presented in Abramsky's "Computational Interpretation of Linear Logic" (1993), it leads to calculus that does **not** satisfy substitution.

Given proofs $\dfrac{\dfrac{\Box A_1 \quad \Box A_2}{B}}{\Box B}$ and $\dfrac{C \to \Box A_1 \quad C}{\Box A_1}$, should be able to substitute A_1 $\dfrac{\dfrac{C \to \Box A_1 \quad C}{\Box A_1} \quad \Box A_2}{B}$ But a problem, as the promotion rule is not applicable, anymore.

Natural Deduction for CS4

Benton, Bierman, de Paiva and Hyland solved the problem for Linear Logic in TLCA 1993.
Bierman and de Paiva (Amsterdam 1992, journal 2000) used the same solution for modal logic.

The solution builds in the substitutions into the rule as

$$\frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k \quad \Box A_1, \ldots, \Box A_k \vdash B}{\Gamma \vdash \Box B} (\Box I)$$

Prawitz uses a notion of essentially modal subformula to guarantee substitutivity in his monograph.

Natural Deduction for CS4

Usual Intuitionistic ND rules plus:

$$\frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k \quad \Box A_1, \ldots, \Box A_k \vdash B}{\Gamma \vdash \Box B}(\Box I) \qquad \frac{\Gamma \vdash \Box A}{\Gamma \vdash A}(\Box E)$$

$$\frac{\Gamma \vdash \Box A_1, \ldots, \Gamma \vdash \Box A_k, \Gamma \vdash \Diamond B \quad A_1 \ldots A_k, B \vdash \Diamond C}{\Gamma \vdash \Diamond C}(\Diamond E) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}(\Diamond I)$$

CS4: Properties/Theorems

- Axioms satisfy Deduction Thm, are equivalent to sequents,
- Sequents satisfy cut-elimination, sub-formula property
- ND is equivalent to sequents
- ND satisfies normalization, ND assigns λ-terms CH equivalent
- Categorical model: monoidal comonad plus box-strong monad
- Issue with Prawitz formulation: idempotency of comonad not warranted...

Problems with system:

- Impurity of rules?
- Commuting conversions, eek!
- what about other modal logics?

Variation on CS4: Dual Intuitionistic and Modal Logic

Following Linear Logic, can define a dual system for $\Box$-only modal logic. DIML, after Barber and Plotkin's DILL, in ICALP 1998.

$$\Gamma, x\colon A, \Gamma' | \Delta \vdash x_M \colon A \qquad \Gamma | \Delta, x\colon A, \Delta' \vdash x_I \colon A$$

$$\frac{\Gamma |\_ \vdash t\colon A}{\Gamma | \Delta \vdash \Box t\colon \Box A} \ (\Box I) \qquad \frac{\Gamma | \Delta \vdash t_i \colon \Box A_i \ \ \Gamma, x_i\colon A_i | \Delta \vdash u\colon B}{\Gamma | \Delta \vdash \texttt{let}\ t_1, \ldots, t_n\ \texttt{be}\ \Box x_1, \ldots, \Box x_n\ \texttt{in}\ u\colon B}$$

Less 'impurity' on rules, less commuting conversions, but what about $\Diamond$? what about other modal systems?

Constructive K: a difficult one

- Constructive K comes from proof-theoretical intuitions provided by Natural Deduction formulations of logic
- Already CS4 does not satisfy distribution of possibiliity over disjunction: $\Diamond(A \vee B) \cong \Diamond A \vee \Diamond B$ and $\Diamond\bot \cong \bot$

Modal Axioms

$\Box(A \to B) \to (\Box A \to \Box B)$
$\Box(A \to B) \to \Diamond A \to \Diamond B$
$(\Box A \times \Diamond B) \to \Diamond(A \times B)$

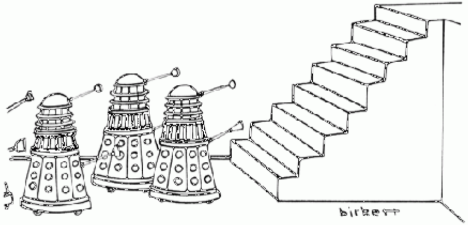
- Sequent rules not as symmetric as in constructive S4, harder to model

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \frac{\Gamma, A \vdash B}{\Box\Gamma, \Diamond A \vdash \Diamond B}$$

- Note: only one rule for each connective, also $\Diamond$ depends on $\Box$.

Constructive K Properties

- Dual-context only for Box fragment
- For Box-fragment, OK. Have subject reduction, normalization and confluence for associated lambda-calculus.
- Have categorical models, but too unconstrained?
- Kripke semantics OK
- No syntax in CK style for Diamonds...
- No ideas for uniformity of systems...
- More work necessary here...



"Well, this certainly buggers our plan to conquer the Universe."

What every one else was doing?

- Simpson: *The Proof Theory and Semantics of Intuitionistic Modal Logic* (1994) a great summary of previous work and a very robust system for geometric theories in Natural Deduction for intuitionistic modal logic
- Intuition from "possible world semantics" interpreted in an intuitionistic metatheory
- Justified by faithfulness of translation into intuitionistic first-order, recovers many of the systems already in the literature
- Strong normalization and confluence proved for all the systems
- Normalization used to establish completeness of cut-free sequent calculi and decidability of some of the systems
- Systems that are decidable also satisfy "finite model property"

What every one else was doing?

- Arnon Avron (1996) Hypersequents (based on Pottinger and Mints)
- Martini and Masini 2-sequents (1996)
- Dosen's higher-order sequents (1985)
- Display calculus (Belnap 1982, Kracht, Gore', survey by Wansing 2002)
- multiple-sequent (more than one kind of sequent arrow) Indrejcazk (1998)
- labelled sequent calculus Negri (2005), inspired by Simpson?
- Nested sequents: Bruennler (2009), Hein, Stewart and Stouppa, Strassburger et al,

What I wanted

- constructive modal logics with axioms, sequents and natural deduction formulations
- Satisfying cut-elimination, finite model property, (strong) normalization, confluence and decidability
- with algebraic, Kripke and categorical semantics
- With translations between formulations and proved equivalences/embeddings
- Translating proofs more than simply theorems
- A broad view of constructive and/or modality
- If possible limitative results

Simpson's desiderata

- IML is a conservative extension of IPL.
- IML contains all substitutions instances of theorems of IPL and is closed under modus ponens.
- If $A \vee B$ is a theorem of IML either A is a theorem or B is a theorem too. (Disjunction Property)
- Box $\Box$ and Diamond $\Diamond$ are independent in IML
- Adding excluded middle to IML yields a standard classical modal logic
- (Intuitionistic) Meaning of the modalities, wrt IML is sound and complete

Avron's desiderata

A generic proof-theoretical framework should:

- Be able to handle a great diversity of logics. Expect to get the ones logicians have used already
- Be independent of any particular semantics
- Structures should be built from formulae in the logic and not too complicated, should yield a "real" subformula property
- Rules of inference should have a small fixed number of premisses, and a local nature of application
- Rules for conjunction, disjunction, implication and negation should be as standard as possible
- Proof systems constructed should give us a better understanding of the corresponding logics and the differences between them

Some divergence: Distribution of Diamond over Disjunction

- distribution of possibility over disjunction binary and nullary: CS4 vs. IS4 (Simpson)

Example (Distribution)

$$\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$$

$$\Diamond\bot \to \bot$$

- This is canonical for classical modal logics
- Many constructive systems don't satisfy it
- Should it be required for constructive ones or not?
- Consequence: adding excluded middle gives you back classical modal logic or not?

Some divergence: labelled vs. unlabelled systems

- proof system should have semantics as part of the syntax?

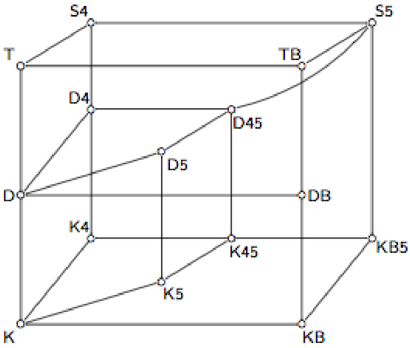
Example (Introduction of Box)

$$\frac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \quad \text{vs.} \quad \frac{\Gamma \quad [xRy] \vdash y : A}{\Gamma \vdash x : \Box A}$$

- The introduction rule for $\Box$ must express that if A holds at every world y visible from x then $\Box A$ holds at x.
- if, on the assumption that y is an arbitrary world visible from x, we can show that A holds at y then we can conclude that $\Box A$ holds at x.
- Simpson's systems have two kinds of hypotheses, $x : A$ which means that the modal formula A is true in the world x and xRy, which says that world y is accessible from world x
- How reasonable is it to have your proposed semantics as part of your syntax?
- Proof-theoretic properties there, but no categorical semantics?

More divergence: modularity of framework?

The framework of ordinary sequents is not capable of handling all interesting logics. There are logics with nice, simple semantics and obvious interest for which no decent, cut-free formulation seems to exist... Larger, but still satisfactory frameworks should, therefore, be sought. Avron (1996)



IK and CK cubes

- Hypersequents, 2-sequents, labelled sequents, nested sequents, display calculi are modular
- Cut-elimination for cubes below, syntax works, but very complicated?...Curry-Howard for CK cube, OK!
- Kripke semantics for CK Mendler and Scheele

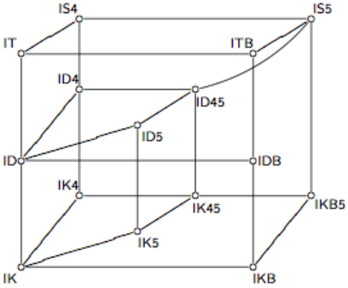
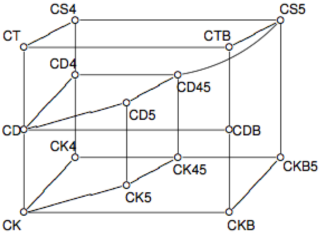


Fig. 2. The intuitionistic "modal cube"



Thanks Lutz Strassburger for pictures!

PreliminarySummary

- Constructive modal logics are interesting for programmers, logicians and philosophers. Shame they don't talk to each other.
- At least two families CK and IK, different properties. Hard to produce good proof theory for them: many augmentations of sequent systems. S5 (classical or intuitionistic) main example
- So far IK better for model theory, CK better for lambda-calculus, but want both, plus categorical semantics too

Can category theory help more?

Categorical models of CK are too unconstrained, not helpful.
Sometimes we can make progress by asking more complicated questions.
Can we do this here?
Modal (dependent) type theory to the rescue
(joint work with Eike Ritter, Birmingham, UK)

Dependent Type Theory

(from Wikipedia) A dependent type is a type that depends on a value. It is an overlapping feature of type theory and type systems. In intuitionistic type theory, dependent types are used to encode logic's quantifiers like "for all" and "there exists". In functional programming languages like ATS, Agda, Idris and Epigram, dependent types prevent bugs by allowing very expressive types.

Renewed interest in the notions of constructive modal type theory and linear type theory? Hope so!

Modal Dependent Type Theory?

Using the Curry-Howard correspondence to each (constructive) modal logic we can associate a modal type theory.

A. Old work on this, following the work on linear type theory, Bierman and de Paiva, Ghani et al modal type theory, no dependent types.

B. Old work of Ritter on extended calculus of constructions, with categorical semantics. no modal types.

New work: Put A and B together for "easy" modal dependent type theory. (easy if you like fibrations, normalization and S4-features)

Dual Modal Intuitionistic Type Theory DIML

(Ghani at al, ICALP 1998), modifying linear calculus of Barber and Plotkin, Benton.

$$\Gamma, x\colon A, \Gamma' | \Delta \vdash x_M\colon A \qquad \Gamma | \Delta, x\colon A, \Delta' \vdash x_I\colon A$$

$$\frac{\Gamma|\Delta, x\colon A \vdash t\colon B}{\Gamma|\Delta \vdash \lambda x\colon A.t\colon A \to B} \qquad \frac{\Gamma|\Delta \vdash t\colon A \to B \quad \Gamma|\Delta \vdash u\colon A}{\Gamma|\Delta \vdash tu\colon B}$$

$$\frac{\Gamma|\_ \vdash t\colon A}{\Gamma|\Delta \vdash \Box t\colon \Box A} \qquad \frac{\Gamma|\Delta \vdash t_i\colon \Box A_i \; \Gamma, x_i\colon A_i|\Delta \vdash u\colon B}{\Gamma|\Delta \vdash \mathtt{let}\ t_1, \ldots, t_n\ \mathtt{be}\ \Box x_1, \ldots, \Box x_n\ \mathtt{in}\ u\colon}$$

$$\frac{\Gamma|\Delta, x\colon A \vdash t\colon B \quad \Gamma|\Delta \vdash u\colon A}{\Gamma|\Delta \vdash (\lambda x\colon A.t)u = t[u/x]\colon B} \qquad \frac{\Gamma|\Delta \vdash t\colon A \to B}{\Gamma|\Delta \vdash \lambda x\colon A.tx = t\colon A \to B}\; x \notin FV$$

Properties of DIML

The introduction of a necessary $\Box$ type requires an empty intuitionistic context. (intuitive idea that a proposition is necessary, if all the assumptions it depends on are already necessary.)
The elimination rule for the $\Box$ modality is traditional, Schroeder-Heister style.
Traditional 1-categorical semantics for DIML easy to state and prove sound and complete.
Type theory was originally (in the linear logic case) derived from the model.
The intuition is that one category S models the modal contexts and modal terms, and the other C the intuitionistic contexts and intuitionistic terms.
The adjunction is used to model the $\Box$-modality.

Theorem 1: Curry-Howard works as expected for this fragment!!

Basic Dependent Type Theory

[ht]

$$\frac{}{[] \vdash \texttt{Context}} \qquad \frac{\Gamma \vdash \texttt{Context} \quad \Gamma \vdash A\ \texttt{Type}}{\Gamma, x : A \vdash \texttt{Context}}$$

$$\frac{\Gamma, x_M : A \vdash B\ \texttt{Type}}{\Gamma \vdash \Pi_{x_M : A} B\ \texttt{Type}}$$

$$\frac{}{\Gamma, x : A, \Gamma' \vdash x : A}$$

$$\frac{\Gamma x : A \vdash t : B}{\Gamma \vdash \lambda x : A.t : \Pi_{x : A} B}\ (\to I) \qquad \frac{\Gamma \vdash t : \Pi_{x : A} B \quad \Gamma \vdash u : A}{\Gamma \vdash tu : B[u/x]}\ (\to E)$$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x : A.t)u = t[u/x] : B[u/x]} \qquad \frac{\Gamma \vdash t : \Pi_{x : A} B}{\Gamma \vdash \lambda x : A.tx = t : \Pi_{x : A} B}\ (x \notin F$$

Putting the Two Together

$$\frac{}{[] \vdash \texttt{Context}} \qquad \frac{\Gamma \vdash \texttt{Context} \quad \Gamma \vdash A\ \texttt{Type}}{\Gamma, x\colon A \vdash \texttt{Context}}$$

$$\frac{\Gamma, x_M\colon A \vdash B\ \texttt{Type}}{\Gamma \vdash \Pi_{x_M\colon A} B\ \texttt{Type}} \qquad \frac{\Gamma, x_M\colon A \vdash B\ \texttt{Type}}{\Gamma \vdash \Box_{x_M\colon A} B\ \texttt{Type}}$$

$$\Gamma, x\colon A, \Gamma' | \Delta \vdash x_M\colon A \qquad \Gamma | \Delta, x_I\colon A, \Delta' \vdash x_I\colon A$$

$$\frac{\Gamma | \Delta, x_I\colon A \vdash t\colon B}{\Gamma | \Delta \vdash \lambda x_I\colon A.t\colon A \to B}\ (\to I) \qquad \frac{\Gamma | \Delta \vdash t\colon A \to B \quad \Gamma | \Delta \vdash u\colon A}{\Gamma | \Delta \vdash tu\colon B}\ ($$

$$\frac{\Gamma, x_M\colon A | \Delta \vdash t\colon B}{\Gamma | \Delta \vdash \lambda x_M\colon A.t\colon \Pi_{x\colon A} B}\ (\Pi I) \qquad \frac{\Gamma | \Delta \vdash t\colon \Pi_{x\colon A} B \quad \Gamma |\_ \vdash u\colon A}{\Gamma | \Delta \vdash tu\colon B[u/x_M]}\ ($$

$$\frac{\Gamma |\_ \vdash t\colon A \quad \Gamma | \Delta \vdash s\colon B[t/x_M]}{\Gamma | \Delta \vdash \Box(t, s)\colon \Box_{x_M\colon A} B}\ (\Box I) \qquad \frac{\Gamma | \Delta \vdash t\colon \Box_{x_M\colon A} B \quad \Gamma, x\colon A | \Delta, y\colon}{\Gamma | \Delta \vdash \texttt{let}\ t\ \texttt{be}\ \Box(x, y)\ \texttt{in}}$$

Properties of Dependent Modal Type Theory

The meta-theory of dependent type theories is usually very subtle.
Subject reduction an important theorem.
Luckily can use the same methodology used for standard dependent type theory.
Hard lifting: to model dependent types, done by Ritter on his phd thesis, using Thomas Ehrhardt's D-categories. (could have used Peter Dybjer's cwfs (category with families), we believe)

Novelty here: Adding dependent modalities, different from Pfenning et al, also different from Anders Schack-Nielsen and Carsten Schuermannn.

Schack-Nielsen's PhD about similar type theory for proof search, not normalization. Earlier work on Cervesato et al. More investigation necessary on pros and cons.

Summary

- Constructive modal logics are interesting for programmers, logicians and philosophers. Shame they don't talk to each other.
- At least two families CK and IK, different properties. Hard to produce good proof theory for them: many augmentations of sequent systems. S5 (classical or intuitionistic) main example
- So far IK better for model theory, CK better for lambda-calculus, but want both, plus categorical semantics too
- Further work
  - Comparison with Schack-Nielsen.
  - Dealing with Diamonds
  - Can extend it to CK? I am sure we can do it for Linear Logic, but applications?