Dialectica Comonoids

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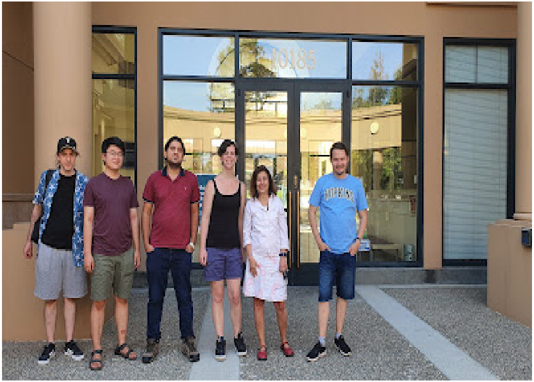
Thank You, Charlotte!



Dialectica?

Dialectica is for Gödel's interpretation

We will not discuss much Proof Theory, we concentrate in Algebra and hence talk about the Dialectica construction and dialectica spaces `https://en.wikipedia.org/wiki/Dialectica_space`.

Dialectica construction?

This is the main subject of this talk, we will define it properly soon. But think about taking two objects of $\mathbb{C}$, U and X and a relation between them, $\alpha : U \times X \to 2$, and imagine a neat notion of morphism between such objects (U, X, α).

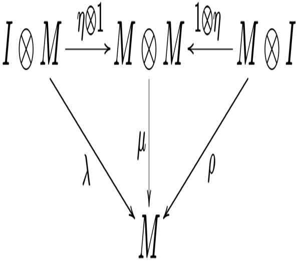
Meanwhile, think about having a cartesian closed category $\mathbb{C}$ and applying the dialectica construction to it.

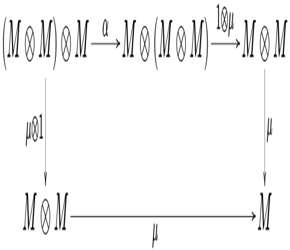
We obtain the dialectica category Dial($\mathbb{C}$) with some interesting logical properties.

Then we want to see what are comonoids in Dial($\mathbb{C}$) and in $\mathbb{C}$ and what they can tell us.

Comonoids?

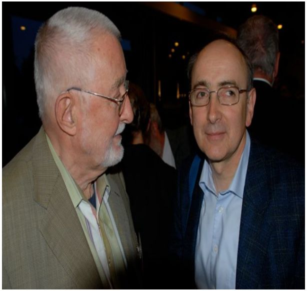
Monoids are very useful and very used.





For comonoids, you just invert all the arrows.
Much less used, so far (it seems to me).
But many applications appearing.
We will discuss one coming from categorical logic.

Categorical logic

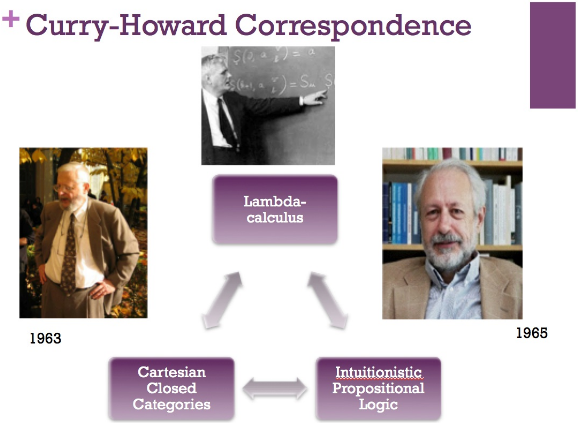


Elegant mathematics will of itself tell a tale, and one with the merit of simplicity. This may carry philosophical weight. But that cannot be guaranteed: in the end one cannot escape the need to form a judgement of significance.

Martin Hyland, *Proof Theory in the Abstract*, 2002. (Kleisli in the picture)

Constructive reasoning

- Why: Reasoning principles that are safer
  - if I ask you whether "is there an x such that $P(x)$?"
  - I'm happier with the answer "yes, x_0", than with an answer "yes, for all x it is not the case that not $P(x)$"
  - want reasoning to be as precise and safe as possible
- How: constructive reasoning as much as possible, classical if need be, but tell me where and why
- Today: monoids, comonoids and (co)monads in categorical logic


+ Curry-Howard Correspondence
Lambda-calculus
1963
1965
Cartesian Closed Categories
Intuitionistic Propositional Logic

Basic Case of Categorical logic

Attach lambda-terms to Natural Deduction proofs and think of them as morphisms in a Cartesian closed category:

$$\frac{\Gamma, x\colon A \vdash t\colon B}{\Gamma \vdash \lambda x\colon A.t\colon A \to B}\ (\to I)$$

$$\frac{\Gamma \vdash t\colon A \to B \quad \Gamma \vdash u\colon A}{\Gamma \vdash tu\colon B}\ (\to E)$$

Logical implication $(\to)$ is the internal-hom (or exponential object) and conjunction is the product $\times$ in the cartesian closed category. The categorical adjunction corresponds to the Deduction theorem

$$A \times B \to C \text{ iff } A \to (B \to C)$$

Comonoids in CCCs

For each object A in $\mathbb{C}$ we have an identity morphism $id_A : A \to A$.

When $\mathbb{C}$ is a CCC, we have a diagonal map $\Delta : A \to A \times A$ and co-unit maps $\rho : A \to A \times 1$ and $\lambda : A \to 1 \times A$ with commuting diagrams.
so every object is a comonoid with respect to the product.
Thus in cartesian closed categories comonoids are not that interesting, as every object is a comonoid, naturally.

This comonoid structure corresponds to the satisfaction of the structural rules of weakening and contraction:

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

Linear Logic and Modalities

- Girard's idea: remove contraction and weakening from usual rules of logic, so logic becomes resource-aware.
- This is GOOD! easier to model features of the world
- But the logic is too weak: to get back the expressive power of usual logic use a modality, written as ! (read as "of course!")
- Contraction and weakening available only for $!A$ formulas/objects.
- ! or bang is a unary operator over a linear basis whose rules correspond to the $\Box$ (or necessity) S4 modality
- It was realized very early on that this modality should be a comonad categorically. Why? Which kind of comonad?

Modalities

Before embarking on details, here is one general piece of advice.

> *One often hears that modal (or some other) logic is pointless because it can be translated into some simpler language in a first-order way. Take no notice of such arguments. There is no weight to the claim that the original system must therefore be replaced by the new one. What is essential is to* single out important concepts *and to investigate their properties. [Scott, 1971]*

Modalities as algebra

Basic idea: Modalities are unary operators over a logic basis. Many logical bases possible: here linear, intuitionistic or classic logic. Operators can be:

$$\text{constructive } \wedge, \vee, \rightarrow, \neg$$

$$\text{or linear ones } \multimap, \parr, \otimes$$

- Which modalities are useful? Which basis?
- **Why? How?**
- Why so many modalities? How to choose?
- Which are the important theorems?
- Which are the most useful applications?

Today: using modalities as motivation for comonoids

Modality Rules

Sequent rules for the modality !

$$\frac{!\Gamma \vdash B}{!\Gamma \vdash !B} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

Reading them as morphisms in a category:

$$\frac{\Gamma = !A, B = !A}{\delta \colon !A \to !!A} \qquad \frac{\Gamma = \emptyset, B = 1}{\mathrm{er} \colon !\mathrm{A} \to 1} \qquad \frac{\Gamma = \emptyset, B = A}{\mathrm{der} \colon !\mathrm{A} \to \mathrm{A}} \qquad \frac{\emptyset, B = !A \otimes !A}{\mathrm{copy} \colon !\mathrm{A} \to !\mathrm{A} \otimes !A}$$

Linear Logic Bang !

Basic idea: each $!A$ needs to be a comonoid with respect to the tensor product. That means it has natural maps:

$$!A \xrightarrow{\text{copy}} !A \otimes !A$$

$$!A \xrightarrow{\text{counit}} I$$

satisfying comonoid requirements
Less obviously $!A$ also needs a coalgebra structure:

$$!A \xrightarrow{\text{der}} A$$

$$!A \xrightarrow{\text{prom}} !!A$$

These structures need to interact in a consistent and coherent way.

Dialectica construction

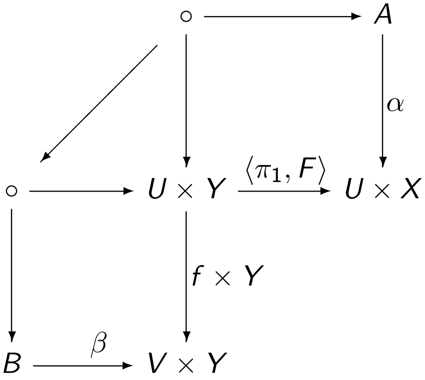
- Dialectica categories are being discussed, because lenses, containers and polynomials look like it
- Most of the conversation has been about the Dialectica morphisms and the symmetric monoidal structure
- But the hard work on Dialectica was proving that it has appropriate comonoids/comonads
- This story is worth telling as it seems to generalize in interesting ways

Dialectica Categories

- A precise model of Linear Logic
- All connectives correspond to distinct structures in the categories
- Two families of models:
  - for ILL original Dialectica categories $\mathrm{Dial}_2(C)$
  - for CLL/FILL Dialectica-like categories $\mathrm{DDial}_2(C)$
- Difference is mostly on morphisms, objects are the same, i.e. triples $(U, X, \alpha\colon A \hookrightarrow U \times X)$ where U, X are objects of C cartesian closed
- Different notions of morphism $\Rightarrow$ different structures in the cats
- Both dialectica cats $\mathrm{Dial}_2(C)$ and $\mathrm{DDial}_2(C)$ are symmetric monoidal closed categories with products.

Original Dialectica Categories

A map in $\mathrm{Dial}_2(C)$ between objects $A = (U, X, \alpha)$ and $B = (V, Y, \beta)$ is a pair of maps in C, $(f\colon U \to V, F\colon U \times Y \to X)$ satisfying the pullback condition below



Dialectica Categories 1

Theorem: (de Paiva 1987) [Linear structure]

The category $\mathrm{Dial}_2(C)$ has a symmetric monoidal closed structure (and products, weak coproducts), that makes it a model of (exponential-free) **intuitionistic** linear logic.

Back to comonads and comonoids

- Surprisingly the ! operator in $\mathrm{Dial}_2(C)$ is a **cofree** comonad
- Many useful well-known monads like exceptions, powerset, continuations, etc... Fewer known and loved comonads
- In our case a comonoid is simply some object that is a monoid in $\mathbb{C}$ in the second coordinate, this defines a comonad in $\mathrm{Dial}(\mathbb{C})$.
- Take $!(U, X, \alpha) = (U, X^*, \alpha*)$, where $(-)^*$ is the free commutative monoid in C.

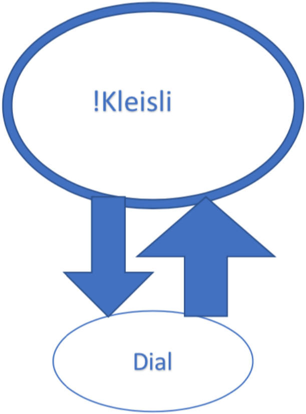
Theorem: [linear and usual logic together]

The **monoidal** comonad ! in $\mathrm{Dial}_2(C)$ above models modalities and recovers Intuitionistic (and Classical) Logic.

Free comonad in $\mathrm{Dial}_2(C)$

- First cofree comonad model, not purely syntactic
- Tensor product is easy $(U, X, \alpha) \otimes (V, Y, \beta)$ is $(U \times V, X \times Y, \alpha \times \beta)$
- ! is a monoidal comonad. There is a natural transformation $m_{(-,-)} : !A \otimes !B \to !(A \otimes B)$ and a morphism $M_I : I \to !I$ satisfying many commutative diagrams
- ! induces a commutative comonoid structure on $!A$
- $!A$ also has naturally a coalgebra structure induced by the comonad !
- The comonoid and coalgebra structures interact in a nice way.

A picture



Comonoids in Dialectica

The approach so far has been:

- We wanted to model a logical system, Linear Logic
- We know what is necessary for a categorical model $\Rightarrow$ objects of a SMCC with both a comonoid and a coalgebra structure (wrt a tensor product and a comonad) interacting nicely
- We prove that we can provide one example of comonoids and coalgebras interacting nicely in the specific case of the Dialectica construction Dial($\mathbb{C}$)

What if we change the approach?

- We have a Dialectica construction, applied to a cartesian closed category $\mathbb{C}$ with finite limits and commutative monoids
- We know Dial($\mathbb{C}$) has a notion of tensor product $\otimes$ which is easy to calculate
- We can describe the category of comonoids for this notion of tensor product
- We know there is at least one object in Dial($\mathbb{C}$) which is a comonoid, the object $(1, 1, id_1)$.
- What are the other comonoids in Dial($\mathbb{C}$)?
- What can we say about the category $Comon_{\otimes}(\mathbb{C})$?

Comonoids and coalgebras in Math

- Barr JPAA 1973 "Coalgebras Over a Commutative Ring" about K-modules that have a comonoid structure
- Hans Porst on local presentability of cats of coalgebras
- Agore on limits of algebras, coalgebras and Hopf algebras

a warning: Sometimes "cocommutative coalgebra" in a symmetric monoidal category is used as a synonym for cocommutative comonoid object.

https://ncatlab.org/nlab/show/cocommutative+coalgebra

Our terminology comes from MacLane's CWM, adding cos whenever necessary.

More comonoids and coalgebras in Math

From https://ncatlab.org/nlab/show/coalgebra+over+a+comonad

Related concepts:

- partial differential equations are the coalgebras of a jet comonad
- well-behaved lenses (in computer science) are the coalgebras of the costate comonad
- model category-structures on coalgebras over a comonad, Hess and Shipley "The homotopy theory of coalgebras over a comonad" arXiv:1205.3979

Two specific examples

- Vaughan Pratt, 2003
  http://boole.stanford.edu/pub/comonoids.pdf, "Comonoids in chu: a large cartesian closed sibling of topological spaces"
- Niu and Spivak, arXiv:2112.11518v3, "Collectives: Compositional protocols for contributions and returns", comonoids in Poly.

Both Chu and Poly are constructions similar to Dialectica.
Chu has the same objects, different morphisms, equalities instead of implications.
Poly has objects that are a dependent version of Dialectica objects, but morphisms as in Chu.

Conclusions

- I promised you a story of comonoids in the Dialectica construction
- I could have called these comonoids 'Linear Modalities' as the (co)monad that introduces them behaves like an S4 modality when considered as logic
- I showed you how dialectica cats introduce several different (co)monads useful to provide models of LL
- I have not talked about constructive modal linear logics
- Constructive modal logics are interesting for programmers, logicians and philosophers. Shame they don't talk much to each other.
- There's much to say on other applications of comonoids, especially in programming languages
- But also on categorical ways of thinking of differentiation!

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